Olimpiada de Matematică –etapa locală- Galaţi

17 februarie 2018

Clasa a XII-a

Barem de evaluare

- **Pentru orice soluţie corectă, chiar dacă este diferită de cea din barem, se acordă punctajul maxim corespunzător.**
- **Nu se acordă fracţiuni de punct, dar se pot acorda punctaje intermediare pentru rezolvări parţiale, în limitele punctajului indicat în barem.**

Nr. problemei	**Soluţie, rezolvare**	**Punctaj**
1.	$f(x)=e^x\cdot\left[2\cdot\sqrt{tg^n x}+n\cdot\sqrt{tg^{n-2}x}\cdot\left(tg^2x+1\right)\right]$.	
	Dar $\left(\sqrt{tg^n x}\right)'=\frac{n}{2}\cdot\sqrt{tg^{n-2}x}\cdot\left(tg^2x+1\right)$.	3p
	Deci $f(x)=2\cdot e^x\cdot\left[\sqrt{tg^n x}+\left(\sqrt{tg^n x}\right)'\right]=2\cdot\left(e^x\cdot\sqrt{tg^n x}\right)'\Rightarrow$	2p
	$\int f(x)dx=2\cdot e^x\cdot\sqrt{tg^n x}+C$.	2p

2.

a) $t \in \mathrm{G} \Leftrightarrow t \in \left(-\infty, \frac{k^2-1}{k}\right] \cup \left[\frac{k^2+1}{k}, \infty\right) \Leftrightarrow$

$t-k \in \left(-\infty, \frac{-1}{k}\right] \cup \left[\frac{1}{k}, \infty\right) \Leftrightarrow |t-k| \geq \frac{1}{k}.$

1p

b)Fie $x, y \in \mathrm{G} \Rightarrow |x-k| \geq \frac{1}{k}$ şi $|y-k| \geq \frac{1}{k} \Rightarrow$

$\left|(x-k)\cdot(y-k)\right| \geq \frac{1}{k^2} \Rightarrow \left|x\cdot y-k\cdot(x+y)+k^2\right| \geq \frac{1}{k^2} \Rightarrow$

$\left|k\cdot x\cdot y-k^2\cdot(x+y)+k^3\right| \geq \frac{1}{k} \Rightarrow \left|k\cdot x\cdot y-k^2\cdot(x+y)+k^3+k-k\right| \geq \frac{1}{k} \Rightarrow$

$|x*y-k| \geq \frac{1}{k} \Rightarrow x*y \in G.$

2p

c) Fie $e \in \mathrm{G}$ astfel încât $x*e=e*x=x,\ (\forall)\, x \in G;$

$x*e=x \Leftrightarrow k\cdot x\cdot e-k^2\cdot(x+e)+k^3+k=x \Leftrightarrow$

$x\cdot\left(k\cdot e-k^2-1\right)-k\cdot\left(k\cdot e-k^2-1\right)=0 \Leftrightarrow$

$(x-k)\cdot\left(k\cdot e-k^2-1\right)=0,\ (\forall)\, x \in G \Rightarrow$

$k\cdot e-k^2-1=0 \Rightarrow e=k+\frac{1}{k} \in G.$

2p

Cum legea "$*$" este comutativă $\Rightarrow e=k+\frac{1}{k}$ este element neutru.

Fie $x \in \mathrm{G}$ şi x' astfel încât $x*x'=x'*x=e.$

$\mathrm{x}*\mathrm{x}'=e \Leftrightarrow k\cdot x\cdot x'-k^2\cdot(x+x')+k^3+k=k+\frac{1}{k} \Rightarrow$

$k\cdot x\cdot x'-k^2\cdot(x+x')+k^3=\frac{1}{k} \Rightarrow k^2\cdot x\cdot x'-k^3\cdot(x+x')+k^4-1=0 \Rightarrow$

$x'\cdot\left(k^2\cdot x-k^3\right)=k^3\cdot x-k^4+1 \Rightarrow$

$x'\cdot k^2\cdot(x-k)=k^3\cdot x-k^4+1 \overset{x\neq k}{\Rightarrow}$

$x'=\frac{k^3\cdot x-k^4+1}{k^2\cdot(x-k)} \Rightarrow x'-k=\frac{k^3\cdot x-k^4+1}{k^2\cdot(x-k)}-k=\frac{1}{k^2\cdot(x-k)};$

Pentru ca $\mathrm{x}' \in \mathrm{G} \Leftrightarrow |x'-k| \geq \frac{1}{k} \Leftrightarrow \frac{1}{\left|k^2\cdot(x-k)\right|} \geq \frac{1}{k} \Leftrightarrow$

$\frac{1}{|x-k|} \geq k \Leftrightarrow |x-k| \leq \frac{1}{k};$

2p

$\left.\begin{array}{l} |x-k| \leq \frac{1}{k} \\ \text{Dar } |x-k| \geq \frac{1}{k} \end{array}\right\} \Rightarrow |x-k|=\frac{1}{k} \Rightarrow$

$x=k\pm\frac{1}{k} \in \mathrm{G}$ sunt singurele elemente simetrizabile.

3.	a) $I_n = \int \frac{2\cdot x^3+3\cdot x^2+5\cdot x+2}{\left(x^2+x+1\right)^n}\,dx = \int \frac{(2\cdot x+1)\left(x^2+x+2\right)}{\left(x^2+x+1\right)^n}\,dx$ Facem substituția $x^2+x+1=t$ $I'_n = \int \frac{t+1}{t^n}dt.$ $n{=}1 \Rightarrow \mathrm{I}'_1 = \int \frac{t+1}{t}dt = \int\left(1+\frac{1}{t}\right)dt = t+\ln t+C \Rightarrow$ $\mathrm{I}_1 = x^2+x+1+\ln\left(x^2+x+1\right)+C;$ $n=2 \Rightarrow \mathrm{I}'_2 = \int \frac{t+1}{t^2}dt = \int\left(\frac{1}{t}+t^{-2}\right)dt = \ln t-\frac{1}{t}+C \Rightarrow$ $\mathrm{I}_2 = \ln\left(x^2+x+1\right)-\frac{1}{x^2+x+1}+C;$ Pentru n $\geq 3 \Rightarrow I'_n = \int \frac{t+1}{t^n}dt = \int \frac{1}{t^{n-1}}dt + \int \frac{1}{t^n}dt = \frac{t^{-n+2}}{-n+2}+\frac{t^{-n+1}}{-n+1}+C =$ $-\frac{1}{(n-2)\cdot t^{n-2}}-\frac{1}{(n-1)\cdot t^{n-1}}+C \Rightarrow$ $\mathrm{I}_n = -\frac{1}{(n-2)\cdot\left(x^2+x+1\right)^{n-2}}-\frac{1}{(n-1)\cdot\left(x^2+x+1\right)^{n-1}}+C,\ n\in\mathbb{N},\ n\geq 3.$	1p 1p 1p
	b) $\int_0^1 \frac{k\cdot\ln a}{a^x+k}dx = k\cdot\int_0^1 \frac{\left(a^x\right)'}{a^x\cdot\left(a^x+k\right)}dx = \int_0^1 \frac{k}{a^x\cdot\left(a^x+k\right)}\cdot\left(a^x\right)'dx =$ $\int_0^1 \frac{a^x+k-a^x}{a^x\cdot\left(a^x+k\right)}\cdot\left(a^x\right)'dx == \int_0^1\left(\frac{1}{a^x}-\frac{1}{a^x+k}\right)\cdot\left(a^x\right)' =$ $= \int_0^1 \frac{\left(a^x\right)'}{a^x}dx - \int_0^1 \frac{\left(a^x+k\right)'}{a^x+k}dx = \ln\left(a^x\right)\Big\vert_0^1 - \ln\left(a^x+k\right)\Big\vert_0^1 = \ln\frac{a\cdot(1+k)}{a+k}.$ $\sum_{k=1}^n \int_0^1 \frac{k\cdot\ln a}{a^x+k}dx = \sum_{k=1}^n \ln\frac{a\cdot(k+1)}{a+k} = \ln\prod_{k=1}^n \frac{a\cdot(k+1)}{a+k} = \ln\frac{a^n\cdot(n+1)!}{(a+1)\cdot(a+2)\cdot\ldots\cdot(a+n)}$ $\sum_{k=1}^n \int_0^1 \frac{k\cdot\ln a}{a^x+k}dx - \ln\left[(n+1)!\right] = \ln\frac{a^n}{(a+1)\cdot(a+2)\cdot\ldots\cdot(a+n)} = x_n,$ Dar $\frac{a^n}{(a+1)\cdot(a+2)\cdot\ldots\cdot(a+n)} < \left(\frac{a}{a+1}\right)^n \to 0 \Rightarrow \lim_{n\to\infty} x_n = -\infty.$	2p 2p

4.	Fie $A_x = \begin{pmatrix} x & f(x) \\ 0 & 1 \end{pmatrix},\ x \in \mathbb{R}^*,\ A_y = \begin{pmatrix} y & f(y) \\ 0 & 1 \end{pmatrix},\ y \in \mathbb{R}^*$; $A_x \cdot A_y = \begin{pmatrix} x & f(x) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y & f(y) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x \cdot y & x \cdot f(y) + f(x) \\ 0 & 1 \end{pmatrix}$ Cum $A_x \cdot A_y \in G$, deducem că $f(x \cdot y) = x \cdot f(y) + f(x)$;	2p
	Dar $A_x \cdot A_y = A_y \cdot A_x \Rightarrow x \cdot f(y) + f(x) = y \cdot f(x) + f(y) \Rightarrow$ $f(x) \cdot (y-1) = f(y) \cdot (x-1)$, Dacă luăm y=2 $\Rightarrow f(x) = f(2) \cdot (x-1)$. Notăm $f(2) = a \in \mathbb{R} \Rightarrow$ $f(x) = a \cdot (x-1)$;	1p
	Atunci $f(x \cdot y) = a \cdot (x \cdot y - 1)(1)$ $x \cdot f(y) + f(x) = a \cdot x \cdot (y-1) + a \cdot (x-1) = a \cdot (x \cdot y - 1)(2)$ Din 1,2 $\Rightarrow A_x \cdot A_y \in G$ $A_x \cdot A_y = A_{x \cdot y} \Rightarrow A_1$ este element neutru; $A_1 = I_2$ $A_x \cdot A_{\frac{1}{x}} = A_1 \Rightarrow$ orice $A_x, x \neq 0$, este simetrizabilă.	2p
	Înseamnă că $(G, \cdot)$ este grup abelian pentru funcţiile $f: \mathbb{R}^* \to \mathbb{R},\ f(x) = a \cdot (x-1),\ a \in \mathbb{R}$.	2p