CONCURSUL NAŢIONAL DE MATEMATICĂ APLICATĂ
“ADOLF HAIMOVICI”

ETAPA LOCALĂ
Suceava, 19 februarie 2017

Clasa a X-a
profil tehnic, profil servicii şi resurse naturale şi protecţia mediului, profil real-specializarea ştiinţele naturii

BAREM DE CORECTARE

1. Fie $z_1, z_2, z_3 \in \mathbb{C}^*$ astfel încât $|z_1| = |z_2| = |z_3| = r$, $r > 0$ şi $z_1 + 3z_2 - 4z_3 = 0$.

a) **(5p)** Arătați că $\frac{1}{z_1} + \frac{3}{z_2} - \frac{4}{z_3} = 0$;

b) **(2p)** Arătați că $z_2 \cdot z_3 + 3z_1 \cdot z_3 - 4z_1 \cdot z_2 = 0$.

Rezolvare: a) Din $|z_1| = |z_2| = |z_3| = r \Rightarrow |z_1|^2 = |z_2|^2 = |z_3|^2 = r^2 \Rightarrow \overline{z_1} = \frac{r^2}{z_1}, \overline{z_2} = \frac{r^2}{z_2}, \overline{z_3} = \frac{r^2}{z_3}$

și cum $z_1 + 3z_2 - 4z_3 = 0 \Rightarrow \overline{z_1} + 3\overline{z_2} - 4\overline{z_3} = 0 \Rightarrow \frac{r^2}{z_1} + \frac{3r^2}{z_2} - \frac{4r^2}{z_3} = 0 \Rightarrow \frac{1}{z_1} + \frac{3}{z_2} - \frac{4}{z_3} = 0$

b) $\frac{1}{z_1} + \frac{3}{z_2} - \frac{4}{z_3} = 0 \Rightarrow \frac{z_2 z_3 + 3z_1 z_3 - 4z_1 z_2}{z_1 z_2 z_3} = 0 \Rightarrow z_2 z_3 + 3z_1 z_3 - 4z_1 z_2 = 0$.

a) $\lvert z_1\rvert = \lvert z_2\rvert = \lvert z_3\rvert = r \Rightarrow \lvert z_1\rvert^2 = \lvert z_2\rvert^2 = \lvert z_3\rvert^2 = r^2 \Rightarrow \overline{z_1} = \frac{r^2}{z_1}, \overline{z_2} = \frac{r^2}{z_2}, \overline{z_3} = \frac{r^2}{z_3}$	2 p
$z_1 + 3z_2 - 4z_3 = 0 \Rightarrow \overline{z_1} + 3\overline{z_2} - 4\overline{z_3} = 0 \Rightarrow \frac{r^2}{z_1} + \frac{3r^2}{z_2} - \frac{4r^2}{z_3} = 0 \Rightarrow \frac{1}{z_1} + \frac{3}{z_2} - \frac{4}{z_3} = 0$	3 p
b) $\frac{1}{z_1} + \frac{3}{z_2} - \frac{4}{z_3} = 0 \Rightarrow \frac{z_2 z_3 + 3z_1 z_3 - 4z_1 z_2}{z_1 z_2 z_3} = 0 \Rightarrow z_2 z_3 + 3z_1 z_3 - 4z_1 z_2 = 0$	2 p

2. Fie a=$\sqrt[3]{\sqrt{2017} + \sqrt{2018}} + \sqrt[3]{\sqrt{2017} - \sqrt{2018}}$.

a) **(3p)** Arătaţi că $a^3 + 3a \in \mathbb{R} \setminus \mathbb{Q}$;

b) **(4p)** Arătaţi că $\log_{2017} \frac{a^3 + 3a}{2} + \log_a \left(\frac{2\sqrt{2017}}{a} - 3 \right) \in \mathbb{Q}$.

Rezolvare:

a) $a^3+3a=\left(\sqrt[3]{\sqrt{2017}+\sqrt{2018}}\right)^3+3\sqrt[3]{\sqrt{2017}+\sqrt{2018}}\cdot\sqrt[3]{\sqrt{2017}-\sqrt{2018}}\cdot$

$\cdot\left(\sqrt[3]{\sqrt{2017}+\sqrt{2018}}+\sqrt[3]{\sqrt{2017}-\sqrt{2018}}\right)+\left(\sqrt[3]{\sqrt{2017}+\sqrt{2018}}\right)^3+3\sqrt[3]{\sqrt{2017}+\sqrt{2018}}+$

$+3\sqrt[3]{\sqrt{2017}-\sqrt{2018}}=\sqrt{2017}+\sqrt{2018}+3\sqrt[3]{2017-2018}\cdot\left(\sqrt[3]{\sqrt{2017}+\sqrt{2018}}+\sqrt[3]{\sqrt{2017}-\sqrt{2018}}\right)+$

$+\sqrt{2017}-\sqrt{2018}+3\sqrt[3]{\sqrt{2017}+\sqrt{2018}}+3\sqrt[3]{\sqrt{2017}-\sqrt{2018}}=$

$=2\sqrt{2017}-3\left(\sqrt[3]{\sqrt{2017}+\sqrt{2018}}+\sqrt[3]{\sqrt{2017}-\sqrt{2018}}\right)+3\left(\sqrt[3]{\sqrt{2017}+\sqrt{2018}}+\sqrt[3]{\sqrt{2017}-\sqrt{2018}}\right)=$

$=2\sqrt{2017}\in\mathbb{R}\setminus\mathbb{Q}$;

b) $\log_{2017}\dfrac{2\sqrt{2017}}{2}+\log_a\left(\dfrac{2\sqrt{2017}}{a}-3\right)=\dfrac{1}{2}+\log_a\dfrac{2\sqrt{2017}-3a}{a}=\dfrac{1}{2}+\log_a\dfrac{a^3}{a}=$

$=\dfrac{1}{2}+\log_a a^2=\dfrac{1}{2}+2=\dfrac{5}{2}\in\mathbb{Q}$.

$\left(a^3+3a=2\sqrt{2017}\Rightarrow 2\sqrt{2017}-3a=a^3\right)$.

a) $a^3+3a=\left(\sqrt[3]{\sqrt{2017}+\sqrt{2018}}\right)^3+3\sqrt[3]{\sqrt{2017}+\sqrt{2018}}\cdot\sqrt[3]{\sqrt{2017}-\sqrt{2018}}\cdot$ $\cdot\left(\sqrt[3]{\sqrt{2017}+\sqrt{2018}}+\sqrt[3]{\sqrt{2017}-\sqrt{2018}}\right)+\left(\sqrt[3]{\sqrt{2017}+\sqrt{2018}}\right)^3+3\sqrt[3]{\sqrt{2017}+\sqrt{2018}}+$ $+3\sqrt[3]{\sqrt{2017}-\sqrt{2018}}$	2 p
$a^3+3a=2\sqrt{2017}\in\mathbb{R}\setminus\mathbb{Q}$	1 p
b) $\log_{2017}\dfrac{2\sqrt{2017}}{2}+\log_a\left(\dfrac{2\sqrt{2017}}{a}-3\right)=\dfrac{1}{2}+\log_a\dfrac{2\sqrt{2017}-3a}{a}=\dfrac{1}{2}+\log_a\dfrac{a^3}{a}$	2 p
Calcule în relația de mai sus și finalizarea $\dfrac{1}{2}+\log_a a^2=\dfrac{1}{2}+2=\dfrac{5}{2}\in\mathbb{Q}$	2 p

3. Fie $E(x,n)=\dfrac{1}{\log_x 2\cdot\log_x 4}+\dfrac{1}{\log_x 4\cdot\log_x 8}+...+\dfrac{1}{\log_x 2^n\cdot\log_x 2^{n+1}},n\in\mathbb{N}^*,x\in(0,1)$.

a) **(4p)** Arătaţi că $E\left(\dfrac{1}{2},n\right)=\dfrac{n}{n+1}$;

b) **(3p)** Arătaţi că $E\left(\frac{1}{2},1\right)\cdot E\left(\frac{1}{2},2\right)\cdot\ldots\cdot E\left(\frac{1}{2},2017\right)=\frac{1}{2018}$.

Rezolvare:

a)

$$E\left(\frac{1}{2},n\right)=\frac{1}{\log_{\frac{1}{2}}2\cdot\log_{\frac{1}{2}}4}+\frac{1}{\log_{\frac{1}{2}}4\cdot\log_{\frac{1}{2}}8}+\ldots+\frac{1}{\log_{\frac{1}{2}}2^n\cdot\log_{\frac{1}{2}}2^{n+1}}$$

$$E\left(\frac{1}{2},n\right)=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n\cdot(n+1)}=\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{n+1}=\frac{n}{n+1}$$

b) $$E\left(\frac{1}{2},n\right)=\frac{n}{n+1}\Rightarrow E\left(\frac{1}{2},1\right)\cdot E\left(\frac{1}{2},2\right)\cdot\ldots\cdot E\left(\frac{1}{2},2017\right)=\frac{1}{2}\cdot\frac{2}{3}\cdot\ldots\cdot\frac{2017}{2018}=\frac{1}{2018}$$

a) $E\left(\frac{1}{2},n\right)=\frac{1}{\log_{\frac{1}{2}}2\cdot\log_{\frac{1}{2}}4}+\frac{1}{\log_{\frac{1}{2}}4\cdot\log_{\frac{1}{2}}8}+\ldots+\frac{1}{\log_{\frac{1}{2}}2^n\cdot\log_{\frac{1}{2}}2^{n+1}}$	1 p
$E\left(\frac{1}{2},n\right)=\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\ldots+\frac{1}{n\cdot(n+1)}$	2 p
$\Rightarrow\frac{1}{1}-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\ldots+\frac{1}{n}-\frac{1}{n+1}=\frac{n}{n+1}$	1 p
b) $E\left(\frac{1}{2},n\right)=\frac{n}{n+1}\Rightarrow E\left(\frac{1}{2},1\right)\cdot E\left(\frac{1}{2},2\right)\cdot\ldots\cdot E\left(\frac{1}{2},2017\right)=\frac{1}{2}\cdot\frac{2}{3}\cdot\ldots\cdot\frac{2017}{2018}$	1 p
$\frac{1}{2}\cdot\frac{2}{3}\cdot\ldots\cdot\frac{2017}{2018}=\frac{1}{2018}$	2 p

4. a) **(3p)** Arătaţi că $\sqrt{a}+\sqrt{b}\le\sqrt{2(a+b)},\forall a\in[0,\infty),\forall b\in[0,\infty)$;

b) **(4p)** Arătaţi că $\sqrt[3]{a}+\sqrt[3]{b}\le\sqrt[3]{4(a+b)},\forall a\in[0,\infty),\forall b\in[0,\infty)$.

Rezolvare:

a) $\sqrt{a}+\sqrt{b}\le\sqrt{2(a+b)}\Leftrightarrow a+2\sqrt{ab}+b\le 2(a+b)\Leftrightarrow\sqrt{ab}\le a+b$

$$\left.\begin{array}{l}\sqrt{ab} \le \dfrac{a+b}{2} \\ \text{Cum } \dfrac{a+b}{2} \le a+b\end{array}\right\} \Rightarrow \sqrt{ab} \le a+b$$

b)

$$\sqrt[3]{a}+\sqrt[3]{b} \le \sqrt[3]{4(a+b)} \Leftrightarrow a+b+3\sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \le 4(a+b) \Leftrightarrow$$

$$3\sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \le 3(a+b) \Leftrightarrow \sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \le (a+b) \Leftrightarrow$$

$$\sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \le \left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left(\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}\right)$$

Cum $\sqrt[3]{ab} \le \sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2} \Leftrightarrow 0 \le \left(\sqrt[3]{a}-\sqrt[3]{b}\right)^2$, obținem inegalitatea cerută.

a) $\sqrt{a}+\sqrt{b} \le \sqrt{2(a+b)} \Leftrightarrow a+2\sqrt{ab}+b \le 2(a+b) \Leftrightarrow \sqrt{ab} \le a+b$	2 p
$\left.\begin{array}{l}\sqrt{ab} \le \dfrac{a+b}{2} \\ \dfrac{a+b}{2} \le a+b\end{array}\right\} \Rightarrow \sqrt{ab} \le a+b$	1 p
b) $\sqrt[3]{a}+\sqrt[3]{b} \le \sqrt[3]{4(a+b)} \Leftrightarrow a+b+3\sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \le 4(a+b)$	2p
$3\sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \le 3(a+b) \Leftrightarrow \sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \le (a+b)$	1p
$\sqrt[3]{ab}\left(\sqrt[3]{a}+\sqrt[3]{b}\right) \le \left(\sqrt[3]{a}+\sqrt[3]{b}\right)\left(\sqrt[3]{a^2}-\sqrt[3]{ab}+\sqrt[3]{b^2}\right)$ și finalizarea	1 p