Matematică

Problema 1 *Dacă* $A \in \mathcal{M}_2(\mathbb{R})$,

$$A = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix},$$

şi $n \in \mathbb{N}^\star$, *atunci* $\mathrm{Tr}(A^n) =$

A) $2^{n+1} \sin \frac{n\pi}{3}$ B) $-2^{n+1} \sin \frac{n\pi}{3}$ C) $2^{n+1} \cos \frac{n\pi}{3}$
D) $-2^{n+1} \cos \frac{n\pi}{3}$ E) $2^n \left[\sqrt{3}^n + \left(-\sqrt{3}\right)^n\right]$ F) 0

Problema 2 *Dacă funcţia* $f : \mathbb{R} \to \mathbb{R}$ *are proprietatea* $f(f(x)) = x^2 - x + 1, \forall x \in \mathbb{R}$, *atunci*

A) $f(0) = 0$ B) $f(0) = 1$ C) $f(1) = 0$ D) $f(1) > f(0)$
E) $f(1) < f(0)$ F) $f(0) = -1$

Problema 3 *Se consideră funcţia* $f : \mathbb{R} \to \mathbb{R}$ *definită prin*

$$f(x) = \begin{cases} \sqrt{x+5}, & \text{dacă } x \in \mathbb{Q} \\ 2 + \sqrt[3]{x+1}, & \text{dacă } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}.$$

Dacă S_f *este suma absciselor punctelor în care funcţia* f *este continuă, atunci* $S_f =$

A) -6 B) $-1 - 2\sqrt{17}$ C) $1 + 2\sqrt{17}$ D) 6 E) 10 F) 0

Problema 4

$$\int_1^2 \frac{x+1}{x^3 + 4x^2} dx =$$

A) $\frac{3 \ln \frac{5}{3} - 2}{16}$ B) $\frac{3 \ln \frac{12}{5} - 2}{16}$ C) $\frac{3 \ln \frac{5}{3} + 2}{16}$ D) $\frac{3 \ln \frac{12}{5} + 2}{16}$
E) $\frac{-3 \ln \frac{5}{3} + 2}{16}$ F) $\frac{3 \ln \frac{3}{5} + 2}{16}$

Problema 5 *Dacă* $x_1, x_2, x_3 \in \mathbb{C}$, *cu* $|x_1| < |x_2| < |x_3|$, *sunt soluţiile ecuaţiei*

$$x^3 + 3x^2 + 2x = 0,$$

atunci $x_1 + 2x_2^2 + 3x_3^3 =$

A) -22 B) 26 C) 0 D) 23 E) 3 F) -3

Problema 6 *Dacă* $S_n = 1 - 3 + 3^2 - 3^3 + \cdots + (-1)^n \cdot 3^n$, $n \in \mathbb{N}^\star$, *atunci* $S_n =$

A) $\frac{(-3)^{n+1} - 1}{2}$ B) $\frac{3^{n+1} - 1}{2}$ C) $\frac{1 - 3^{n+1}}{4}$ D) $\frac{1 - (-3)^{n+1}}{2}$
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Problema 7 *Suma soluţiilor reale ale ecuaţiei* $3\lg^2(x) - 2\lg(x) - 1 = 0$ *este*

A) $\frac{2}{3}$ B) $-\frac{2}{3}$ C) $\frac{100+\sqrt[3]{100}}{10}$ D) $10+\sqrt[3]{10}$
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Problema 8

$$\det\begin{pmatrix} 1 & 1 & -1 \\ \sqrt{2} & 0 & -\sqrt{3} \\ -\sqrt{3} & \sqrt{2} & 1 \end{pmatrix} =$$

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Problema 9 *Se consideră inecuaţia*

$$3^{\log_{\frac{1}{2}} x} > 1, x \in \mathbb{R}.$$

Soluţia inecuaţiei este

A) $x \in \emptyset$ B) $x \in (0,1)$ C) $x \in (1,\infty)$ D) $x \in (1,3)$ E) $x \in (0,3)$ F) $x \in \mathbb{R}$

Problema 10 *Dacă* $n \in \mathbb{N}^\star$, *atunci*

$$\lim_{n\to\infty} n \sin\frac{1}{n} =$$

A) ∞ B) 1 C) 0 D) $\frac{1}{2}$ E) nu există F) $\frac{1}{e}$

Matematică - RĂSPUNSURI

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$$\lim_{n\to\infty} n \sin\frac{1}{n} =$$

A) ∞ $\boxed{\text{B) } 1}$ C) 0 D) $\frac{1}{2}$ E) nu există F) $\frac{1}{e}$

Matematică

Problema 1 C) $2^{n+1}cos\frac{n\pi}{3}$

Pentru rezolvarea următoarei probleme vom folosi matricea trigonometrică:

$\begin{pmatrix} cosx & -sinx \\ sinx & cosx \end{pmatrix}$ care are următoarea propietate:

$$\begin{pmatrix} cosx & -sinx \\ sinx & cosx \end{pmatrix}^n = \begin{pmatrix} cos(nx) & -sin(nx) \\ sin(nx) & cos(nx) \end{pmatrix}$$

Pornim de la matricea dată:

$$A = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = 2\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = 2\begin{pmatrix} cos\frac{\pi}{3} & -sin\frac{\pi}{3} \\ sin\frac{\pi}{3} & cos\frac{\pi}{3} \end{pmatrix}$$

Asadar, $A^n = 2^n\begin{pmatrix} cos\frac{n\pi}{3} & -sin\frac{n\pi}{3} \\ sin\frac{n\pi}{3} & cos\frac{n\pi}{3} \end{pmatrix}$

Deci, $Tr(A^n) = 2^n cos\frac{n\pi}{3} + 2^n cos\frac{n\pi}{3} = 2^{n+1}cos\frac{n\pi}{3}$

Problema 2 B) $f(0) = 1$

$f(f(f(x))) = f(x^2 - x + 1) = f(x)^2 - f(x) + 1$

Daca $x = 1$ rezulta din relatia de mai sus: $f(1) = f(1)^2 - f(1) + 1 \Rightarrow$
$\Rightarrow f(1) = 1$

Daca $x = 0$ rezulta: $f(1) = f(0)^2 - f(0) + 1$,iar cum$f(1) = 1 \Rightarrow$

$\Rightarrow f(0)^2 - f(0) = 0 \Rightarrow f(0) = 1$ sau $f(0) = 0$.

Verificam rezultatele in ecuatia initiala si observam ca $f(0) = 0$ nu verifica. Asadar raspunsul final este: $f(0) = 1$

Problema 3 E) 10

Pentru a afla abscisele punctelor de continuitate vom egala ramurile functiei si rezolvam ecuatia:

$\sqrt{x+5} = 2 + \sqrt[3]{x+1}$

Notam

$a = \sqrt{x+5} \Rightarrow 5 + x = a^2 \Rightarrow x = a^2 - 5(1)$

$b = \sqrt[3]{x+1} \Rightarrow x + 1 = b^3 \Rightarrow x = b^3 - 1(2)$

Din (1) si (2) $\Rightarrow a^2 - 5 = b^3 - 1$

Cum $a = b + 2$

$\Rightarrow (b+2)^2 - 5 = b^3 - 1 \Leftrightarrow b^2 + 4b + 4 - 5 = b^3 - 1 \Leftrightarrow b^3 - b^2 - 4b = 0 \Leftrightarrow$

$\Leftrightarrow b(b^2 - b - 4) = 0$

I. $b = 0 \Rightarrow x_1 = -1$

II. $b^2 - b - 4 = 0$

$\Delta = (-1)^2 + 16 = 17$
$b = \dfrac{1 \pm \sqrt{17}}{2} \Leftrightarrow a = \dfrac{5 \pm \sqrt{17}}{2}$

$$x_2 = \left(\frac{5+\sqrt{17}}{2}\right)^2 - 5 = \frac{25+17+10\sqrt{17}-20}{4} = \frac{22+10\sqrt{17}}{4} = \frac{11+5\sqrt{17}}{2}$$

$$x_3 = \left(\frac{5-\sqrt{17}}{2}\right)^2 - 5 = \frac{25+17-10\sqrt{17}-20}{4} = \frac{22-10\sqrt{17}}{4} = \frac{11-5\sqrt{17}}{2}$$

$$S = -1 + \frac{11+5\sqrt{17}}{2} + \frac{11-5\sqrt{17}}{2} = -1 + \frac{22}{2} = -1 + 11 = 10$$

Problema 4 C) $\dfrac{3ln\frac{5}{3}+2}{16}$

$$\int_1^2 \frac{x+1}{x^3+4x^2}dx = \int_1^2 \frac{x+1}{x^2(x+4)}dx$$

Folosim metoda descompunerii in fractii simple.

$$\frac{A}{x+4} + \frac{B}{x} + \frac{C}{x^2} \underset{\text{Aducem la acelasi numitor}}{\equiv} \frac{Ax^2 + B(x(x+4)) + C(x+4)}{x^2(x+4)}$$

$$x+1 = Ax^2 + B(x(x+4)) + C(x+4) = Ax^2 + Bx^2 + 4Bx + Cx + 4C =$$

$$= (A+B)x^2 + (4B+C)x + 4C \Rightarrow$$

$$\Rightarrow \begin{cases} A+B=0 \\ 4B+C=1 \\ 4C=1 \end{cases} \Rightarrow \begin{cases} A+B=0 \\ 4B+C=1 \\ C=\frac{1}{4} \end{cases} \Rightarrow \begin{cases} A+B=0 \\ B=\frac{3}{16} \\ C=\frac{1}{4} \end{cases} \Rightarrow \begin{cases} A=-\frac{3}{16} \\ B=\frac{3}{16} \\ C=\frac{1}{4} \end{cases}$$

$$\int_1^2 \frac{x+1}{x^3+4x^2}dx = \int_1^2 (-\frac{3}{16(x+4)} + \frac{3}{16x} + \frac{1}{4x^2})dx = -\frac{3}{16}\int_1^2 \frac{1}{x+4}dx +$$

$$+\frac{3}{16}\int_1^2 \frac{1}{x}dx + \frac{1}{4}\int_1^2 \frac{1}{x^2}dx = \left(-\frac{3}{16}(ln(x+4) - ln(x)) - \frac{1}{4x}\right)\Bigg|_1^2 =$$

$$= -\frac{3}{16}(ln(6) - ln(2)) - \frac{1}{8} \ - (-\frac{3}{16}(ln(5) - ln(1)) - \frac{1}{4}) =$$

$$= -\frac{3}{16}ln\frac{6}{2} - \frac{2}{16} + \frac{3}{16}ln5 + \frac{4}{16} = \frac{-3ln3 + 3ln5 + 2}{16} = \frac{3ln\frac{5}{3} + 2}{16}$$

Problema 5 A) -22

$x^3 + 3x^2 + 2x = 0 \Leftrightarrow x(x^2 + 3x + 2) = 0$

I. $x_1 = 0$

II. $x^2 + 3x + 2 = 0 \Leftrightarrow (x+1)(x+2) = 0 \Rightarrow$

$x_2 = -1; x_3 = -2;$

Deoarece $|x_1| < |x_2| < |x_3|$

$x_1 + 2x_2^2 + 3x_3^3 \Leftrightarrow 0 + 2*(-1)^2 + 3*(-2)^3 = 2 + 3*(-8) = 2 - 24 = -22$

Problema 6 F) $\frac{1-(-3)^{n+1}}{4}$

Avem o suma de numere in progresie geometrica, cu ratia $q = -3$ si primul element $b_1 = 1$.

Formula sumei progresiei geometrice este: $S_n = \frac{b_1(q^n-1)}{q-1}$.

$$S_n = \frac{1((-3)^{n+1}-1)}{-3-1} = \frac{-1+(-3)^{n+1}}{-4} = \frac{1-(-3)^{n+1}}{4}$$

Problema 7 C) $\dfrac{100+\sqrt[3]{100}}{10}$

Notam y=lg(x)

$3y^2 - 2y - 1 = 0$

$\Delta = b^2 - 4ac = 4 + 12 = 16$

$$y = \frac{-b \pm \sqrt{\Delta}}{2a} = \frac{2 \pm 4}{6}$$

$y_1 = 1; y_2 = -\dfrac{1}{3}$

I. 1=lg(x_1)$\Leftrightarrow$ x_1=10

II. - $\dfrac{1}{3}$=lg(x_2)$\Leftrightarrow$ x_2=$10^{-\frac{1}{3}} = \dfrac{1}{\sqrt[3]{10}} = \dfrac{\sqrt[3]{10^2}}{10}$

S=10+$\dfrac{\sqrt{10^2}}{10}$=$\dfrac{100+\sqrt[3]{10^2}}{10}$

Problema 8 A) $1-\sqrt{2}+\sqrt{6}$

$$\begin{pmatrix} 1 & 1 & -1 \\ \sqrt{2} & 0 & -\sqrt{3} \\ -\sqrt{3} & \sqrt{2} & 1 \end{pmatrix} = 0 + (-2) + 3 - 0 - (-\sqrt{6}) - \sqrt{2} = 1 + \sqrt{6} - \sqrt{2}$$

Problema 9 B) $x \in (0,1)$

Din conditile de existenta ale logaritmului $\Rightarrow x > 0(1)$

Notam a=$\log_{\frac{1}{2}} x$

Inecuatia devine $3^a > 1 \Rightarrow \log_3 3^a > \log_3 1 \Rightarrow a > 0$

Revenim la notatie: $\log_{\frac{1}{2}} x > 0 \Rightarrow \log_{2^{-1}} x > 0 \Rightarrow$

$\Rightarrow -\log_2 x > 0 \Rightarrow \log_2 x < 0 \Rightarrow x < 1(2)$

Din (1) si (2) $\Rightarrow x \in (0,1)$

Problema 10 B) 1

$$\lim_{n\to\infty} nsin\frac{1}{n} = \lim_{n\to\infty} \frac{sin\frac{1}{n}}{\frac{1}{n}} \overset{\text{Limita remarcabila}}{=\!=\!=\!=\!=} 1$$