Barem clasa a VIII-a
(OLM 2018-etapa locală)

Problema I. (7 puncte)

a) Egalitatea dată se poate scrie $x - y\sqrt{2018} = 2018y + 2018\sqrt{2018} \Rightarrow x - 2018y = \sqrt{2018}\cdot(y+2018)$.....**(1p)**

Deoarece $x, y \in \mathrm{Q}$ $\Rightarrow x - 2018y \in$ Q, ceea ce conduce la $y + 2018 = 0 \Rightarrow y = -2018$......................**(1p)**

Înlocuind, se obţine $x - 2018y = 0 \Rightarrow x + 2018^2 = 0 \Rightarrow x = -2018^2$..**(1p)**

b) Folosind inegalitatea mediilor obţinem

$\sqrt{\frac{b+c+d}{a}} = \sqrt{\frac{b+c+d}{a}\cdot 1} \le \frac{1}{2}\left(\frac{b+c+d}{a}+1\right)$, deci $\sqrt{\frac{b+c+d}{a}} \le \frac{a+b+c+d}{2a}$, $\Rightarrow \sqrt{\frac{a}{b+c+d}} \ge \frac{2a}{a+b+c+d}$ ....**(1p)**

Analog obţinem şi $\sqrt{\frac{b}{a+c+d}} \ge \frac{2b}{a+b+c+d}$, $\sqrt{\frac{c}{a+b+d}} \ge \frac{2c}{a+b+c+d}$, $\sqrt{\frac{d}{a+b+c}} \ge \frac{2d}{a+b+c+d}$ ....**(2p)**

Deci $\sqrt{\frac{a}{b+c+d}} + \sqrt{\frac{b}{a+c+d}} + \sqrt{\frac{c}{a+b+d}} + \sqrt{\frac{d}{a+b+c}} \ge \frac{2a+2b+2c+2d}{a+b+c+d}$ =2 ....**(1p)**

Problema II. (7 puncte)

$\frac{x\sqrt{2}+y\sqrt{3}}{y\sqrt{2}+z\sqrt{3}} = \frac{\left(x\sqrt{2}+y\sqrt{3}\right)\left(y\sqrt{2}-z\sqrt{3}\right)}{2y^2-3z^2} = \frac{2xy-3yz+\sqrt{6}\left(y^2-xz\right)}{2y^2-3z^2} \in \mathbf{Q}$..**(3p)**

Se deduce că $y^2 = xz \Rightarrow$ $x^4 + y^4 + z^4 = x^4 + x^2z^2 + z^4$ ....**(2p)**

Așadar, $x^4 + y^4 + z^4 = \left(x^2+z^2\right)^2 - x^2z^2 = (x^2 - xz + z^2)\left(x^2 + xz + z^2\right) = \left(x^2 - y^2 + z^2\right)\left(x^2 + y^2 + z^2\right)$

de unde concluzia.. **(2p)**

Problema III. (7 puncte)

a) $E(n) = \frac{1}{3}\left(\sqrt{3\cdot 1+2} - \sqrt{2} + \sqrt{3\cdot 2+2} - \sqrt{3\cdot 1+2} + \sqrt{3\cdot 3+2} - \sqrt{3\cdot 2+2+} \cdots + \sqrt{3\cdot n+2} - \sqrt{3\cdot n-1}\right) = \frac{\sqrt{3\cdot n+2}-\sqrt{2}}{3}$..**(2p)**

Dacă $E(n) \in \mathbb{Q}$ rezultă că $\sqrt{3\cdot n+2} - \sqrt{2} = k \in \mathbb{N}^*$;

$\sqrt{3\cdot n+2} = k + \sqrt{2}$; $3\cdot n + 2 = k^2 + 2 + 2\sqrt{2}\cdot k$, de unde $\sqrt{2} = \frac{3\cdot n - k^2}{2k}$ imposibil deoarece $\frac{3\cdot n-k^2}{2k} \in \mathbb{Q}$, iar $\sqrt{2} \notin \mathbb{Q}$. Deci $E(n) \in \mathbb{R}\backslash\mathbb{Q}$, oricare ar fi $n \in \mathbb{N}^*$...................................**(2p)**

b) $\frac{E(n)}{\sqrt{n}} = \frac{\sqrt{3 \cdot n+2}-\sqrt{2}}{3 \cdot \sqrt{n}}$. Evident $0 < \frac{E(n)}{\sqrt{n}}$ pentru orice $n \in \mathbb{N}^*$..**(1p)**

Demonstrăm că $\frac{E(n)}{\sqrt{n}} < 1$. Avem $\sqrt{3 \cdot n + 2} - \sqrt{2} < 3 \cdot \sqrt{n}$; $\sqrt{3 \cdot n + 2} < 3 \cdot \sqrt{n} + \sqrt{2}$;

$3 \cdot n + 2 < 9 \cdot n + 6 \cdot \sqrt{2} \cdot \sqrt{n} + 2$; $\quad 0 < 6 \cdot n + 6 \cdot \sqrt{2} \cdot \sqrt{n}$, evident pentru orice $n \in \mathbb{N}^*$........**(1p)**

Astfel, partea întreagă a lui $\frac{E(n)}{\sqrt{n}}$ este 0, oricare ar fi $n \in \mathbb{N}^*$..**(1p)**

Problema IV. (7 puncte)

Desen corect..(1p)

a) ΔABC , $m(\sphericalangle A) = 90°$, BC=100 mm

Construim $AD \perp BC \Rightarrow AD = \frac{AB \cdot AC}{BC} = 48$ mm..**(1p)**

$$\left.\begin{array}{c} MA \perp (ABC) \\ AD \perp BC \\ AD, BC \subset (ABC) \end{array}\right\} \xRightarrow{T\,3\perp} MD \perp BC, MD = 60 \text{ mm}$$ ....**(1p)**

$A_{MBC} = \frac{BC \cdot MD}{2} = 3000\ mm^2$ ....**(1p)**

b) Ducem $AT \perp MD$

$$\left.\begin{array}{c} BC \perp MA \\ BC \perp AD \\ AD \cap MA = \{A\} \end{array}\right\} \Rightarrow \left.\begin{array}{c} BC \perp (MAD) \\ AT \subset (MAD) \end{array}\right\} \Rightarrow BC \perp AT \Rightarrow AT \perp BC$$ ....**(1p)**

$$\left.\begin{array}{c} AT \perp BC \\ AT \perp MD \\ MD \cap BC = \{D\} \end{array}\right\} \Rightarrow AT \perp (MBC) \Rightarrow d\big(A, (MBC)\big) = AT$$ ....**(1p)**

ΔMAD , $m(\sphericalangle A) = 90° \Rightarrow AT = \frac{AM \cdot AD}{MD} = 28{,}8$ mm< 30mm=3cm..**(1p)**