Barem clasa a IX-a
(OLM 2018-etapa locală)

Problema I. (7 puncte)

Aplicând inegalitatea lui Cauchy-Buniakowski-Schwartz, avem

$$\left(a^2+bc\right)\left(1+\frac{b}{c}\right)\geq\left(a\cdot 1+\sqrt{bc}\cdot\sqrt{\frac{b}{c}}\right)^2 \Leftrightarrow \left(a^2+bc\right)\left(1+\frac{b}{c}\right)\geq\left(a+b\right)^2$$ ......**(2p)**

$$\left(b^2+ac\right)\left(1+\frac{c}{a}\right)\geq\left(b\cdot 1+\sqrt{ac}\cdot\sqrt{\frac{c}{a}}\right)^2 \Leftrightarrow \left(b^2+ac\right)\left(1+\frac{c}{a}\right)\geq\left(b+c\right)^2$$ ......**(2p)**

$$\left(c^2+ab\right)\left(1+\frac{a}{b}\right)\geq\left(c\cdot 1+\sqrt{ab}\cdot\sqrt{\frac{a}{b}}\right)^2 \Leftrightarrow \left(c^2+ab\right)\left(1+\frac{a}{b}\right)\geq\left(c+a\right)^2$$ ......**(2p)**

Inmulţind relaţiile între ele se obţine egalitatea cerută......**(1p)**

Problema II. (7 puncte)

$x^2-2xy+2y^2-2ay+4y=4a-a^2$

$(x-y)^2+y^2-2ay+4y=4a-a^2$

$(x-y)^2=-y^2+2ay-4y+4a-a^2$

$(x-y)^2=-\left(y^2-2ay+a^2\right)-4y+4a$ ......**(2p)**

$(x-y)^2=-(y-a)(y-a+4)$

$(x-y)^2\geq 0\Rightarrow (y-a)(y-a+4)\leq 0\Rightarrow$

$$\Rightarrow \left.\begin{matrix} y\in[a,a-4]\text{sau}[\text{a - 4, a}] \\ a,y\in Z\end{matrix}\right\}\Rightarrow y\in\{a-4,a-3,a-2,a-1,a\}$$

1. $y=a-4\Rightarrow (x-y)^2=0\Rightarrow x=y=a-4$ ......**(1p)**

2.$y=a-3\Rightarrow (x-y)^2=-(a-3-a)(a-3-a+4)\Rightarrow (x-y)^2=3\Rightarrow x-y=\pm\sqrt{3}$, imp.$(x,y\in Z)$....**(1p)**

3. $y=a-2\Rightarrow (x-y)^2=-(a-2-a)(a-2-a+4)\Rightarrow (x-y)^2=4\Rightarrow\Rightarrow x-y=\pm 2\Rightarrow x=y\pm 2\Rightarrow x\in\{a,a-4\}$.**(1p)**

4. $y=a-1\Rightarrow (x-y)^2=-(a-1-a)(a-1-a+4)=3\Rightarrow x-y=\pm\sqrt{3}$, imposibil $(x,y\in Z)$ ......**(1p)**

5. $y=a\Rightarrow (x-y)^2=0\Rightarrow x=y=a$ ......**(1p)**

$S=\{(a-4,a-4);(a-4,a-2);(a,a-2);(a,a)\}$

Problema III. (7 puncte)

Avem $\left(\sqrt{5}+\sqrt{3}\right)^{2018}=\left(8+2\sqrt{15}\right)^{1009}$ ....**(2p)**

Arătăm că $\left(8+2\sqrt{15}\right)^{1009}+\left(8-2\sqrt{15}\right)^{1009}$ este număr natural divizibil cu 64. Pentru aceasta, prin inducţie matematică, vom demonstra că $a_n=\left(8+2\sqrt{15}\right)^{2n+1}+\left(8-2\sqrt{15}\right)^{2n+1} \vdots\, 64$ ....**(1p)**

Dacă notăm $x_1=8+2\sqrt{15}$, $x_2=8-2\sqrt{15}$ atunci avem de demonstrat propoziţia

$P(n)$: ${x_1}^{2n+1}+{x_2}^{2n+1} \vdots\, 64$

I. Verificare $P(1)$..**(1p)**

II. Pentru etapa a doua, considerăm ecuaţia de gradul al doilea cu rădăcinile x_1, x_2, $x^2-16x+4=0$, care rezolvă pasul de inducţie. ....**(1p)**

Rezultă $a_{504}=\left(8+2\sqrt{15}\right)^{1009}+\left(8-2\sqrt{15}\right)^{1009} \vdots\, 64$, deci $\left(\sqrt{5}+\sqrt{3}\right)^{2018}=a_{504}-\left(8-2\sqrt{15}\right)^{1009}$ şi cum $\left(8-2\sqrt{15}\right)\in(0,1)$, rezultă că $\left[\left(\sqrt{5}+\sqrt{3}\right)^{2018}\right]=64\cdot p-1,\ p\in N^*$, sau $\left[\left(\sqrt{5}+\sqrt{3}\right)^{2018}\right]=64\cdot(p-1)+63$. Prin urmare restul căutat este 63..**(2p)**

Problema IV. (7 puncte)

Deorece A_1, B_1, C_1 sunt punctele diametral opuse vârfurilor A,B și C , avem că

$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=-(\overrightarrow{OA_1}+\overrightarrow{OB_1}+\overrightarrow{OC_1})$. ....(1p)

Folosind relaţia lui Sylvester în triunghiurile A_1BC, AB_1C respectiv ABC_1, obţinem

$\overrightarrow{OH_1}=\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OA_1}$

$\overrightarrow{OH_2}=\overrightarrow{OA}+\overrightarrow{OC}+\overrightarrow{OB_1}$

$\overrightarrow{OH_3}=\overrightarrow{OB}+\overrightarrow{OA}+\overrightarrow{OC_1}$,de unde prin adunare

avem $\overrightarrow{OH_1}+\overrightarrow{OH_2}+\overrightarrow{OH_3}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$ ....(2p)

H_4, H_5, H_6 fiind simetricele punctelor H_1, H_2, H_3 faţă de vârfurile A_1, B_1, C_1 avem

$\overrightarrow{OA_1}=\dfrac{\overrightarrow{OH_1}+\overrightarrow{OH_4}}{2}$ de unde $\overrightarrow{OH_4}=2\overrightarrow{OA_1}-\overrightarrow{OH_1}$

$\overrightarrow{OB_1}=\dfrac{\overrightarrow{OH_2}+\overrightarrow{OH_5}}{2}$ de unde $\overrightarrow{OH_5}=2\overrightarrow{OB_1}-\overrightarrow{OH_2}$ ....(2p)

$\overrightarrow{OC_1}=\dfrac{\overrightarrow{OH_3}+\overrightarrow{OH_6}}{2}$ de unde $\overrightarrow{OH_6}=2\overrightarrow{OC_1}-\overrightarrow{OH_3}$.

AH_4, BH_5, CH_6 pot fi laturile unui triunghi..(1p)

$\Leftrightarrow \overrightarrow{AH_4}+\overrightarrow{BH_5}+\overrightarrow{CH_6}=\vec{0} \Leftrightarrow \overrightarrow{AO}+\overrightarrow{BO}+\overrightarrow{CO}+\overrightarrow{OH_4}+\overrightarrow{OH_5}+\overrightarrow{OH_6}=\vec{0}$

$\Leftrightarrow 4\left(\overrightarrow{AO}+\overrightarrow{BO}+\overrightarrow{CO}\right)=\vec{0} \Leftrightarrow \overrightarrow{AO}+\overrightarrow{BO}+\overrightarrow{CO}=\vec{0} \Leftrightarrow$ triunghiul ABC echilateral.......(1p)