József Wildt International Mathematical Competition

The Edition XXVIth , 2016

The solution of the problems W.1 - W.50 must be mailed before 10. September 2016, to Mihály Bencze, str. Hărmanului 6, 505600 Săcele - Négyfalu, Jud. Braşov, Romania, E-mail: benczemihaly@yahoo.com; benczemihaly@gmail.com

W1. Show that $H = \sum_{\substack{q \leq Q \\ q \text{ odd, squarefree}}} \prod_{p|q} \frac{2}{p-2} \gg (\log Q)^2$.

Michael Rassias

W2. Let D be a fixed squarefree integer with $D \geq 1$. Let also $\pi_D(x) = \sum_{\substack{p \leq x \\ \left(\frac{-D}{p}\right)=1}} 1$.

Prove that $\pi_D(x) \sim \frac{1}{2}\pi(x)$, when $x \to +\infty$

Michael Rassias

W3. Let $(F_k)_{k \geq 0}, F_0 = 0, F_1 = 1, F_{k+2} = F_k + F_{k+1}$, $(\forall)\, k \in N$ and $A = (a_{ij})_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}}$, $B = (b_{jk})_{\substack{1 \leq j \leq n \\ 1 \leq k \leq m}}$, $C = (c_{rs})_{1 \leq r,s \leq m}$, $a_{ij} = F_j$, $b_{jk} = F_j$, $c_{rs} = F^2_{m-r+1}$, $(\forall)\, i, k = \overline{1, m}$, $(\forall)\, \overline{1, n}$. For p, q positive integers compute $(AB)^p\, C^q$.

D.M.Bătineţu-Giurgiu and Neculai Stanciu

W4. Let $a, b \in (-\infty, +\infty)$. Find $\lim\limits_{n \to \infty} \left(\frac{(n+3)^{n+2+a}}{(n+2)^{n+1+b}} - \frac{(n+2)^{n+1+a}}{(n+1)^{n+b}} \right)$.

D.M.Bătineţu-Giurgiu and Neculai Stanciu

W5. Let $f : R^*_+ \to R^*_+$ be a continue function such that $\lim\limits_{x \to \infty} \frac{f(x)}{x^m} = a \in R^*_+$, where $m \in [1, +\infty)$.

Calculate $\lim\limits_{n \to \infty} \left(\sqrt[n+1]{\prod\limits_{k=1}^{n} f(k)} - \sqrt[n]{\prod\limits_{k=1}^{n} f(k)} \right)$.

D.M.Bătineţu-Giurgiu and Neculai Stanciu

W6. If a, b, c are the sides of a triangle, demonstrate the inequality $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{r}{R} \leq 2$.

Stănescu Florin

W7. If all triangle ABC holds $\sum \sin A - \prod \sin A \geq \sum \sin^3 A \geq \prod \sin A + 4 \prod \cos A \left(\sum \sin A\right)$.

Stănescu Florin

W8. Let $f, g : [0,1] \to (0, +\infty)$, $f(0) = g(0) = 0$ two continuous functions such that f it's convex , and g concave. If $h : [0,1] \to R$ is an increasing function, show that
$\int\limits_0^1 h(x)\, g(x)\, dx \int\limits_0^1 f(x)\, dx \leq \int\limits_0^1 g(x)\, dx \int\limits_0^1 h(x)\, f(x)\, dx.$

Stănescu Florin

W9. Let n be a positive integer. Prove that

$$\sum_{k=1}^{n} \frac{F_{2k}}{(F_{2k+1} - 1)^2} < 2,$$

where F_n is the n^{th} Fibonacci number. That is, $F_0 = 0, F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for $n \geq 1$.

José Luis Díaz-Barrero

W10. Let a, b, and c be positive real numbers. Prove that

$$\left(\frac{(6n+1)a-b}{n(b+c)}\right)^2+\left(\frac{(6n+1)b-c}{n(c+a)}\right)^2+\left(\frac{(6n+1)c-a}{n(a+b)}\right)^2\geq 27$$

for any positive integer $n \geq 1$.

José Luis Díaz-Barrero

W11. Let $x > y > z > t$ be four positive integers such that

$$(x^2-y^2)+(xz-yt)-(z^2-t^2)=0.$$

Prove that $xy+zt$ is a composite number.

José Luis Díaz-Barrero

W12. Let $n \in N$ and let $O_n = 1+\frac{1}{3}+...+\frac{1}{2n-1}$. Calculate $\lim\limits_{n\to\infty}\frac{1}{n}\left(1+\frac{20n}{n}\right)^n$.

Ovidiu Furdui

W13. Let $(a_n)_{n\in N}$ be a sequence of real numbers such that $\lim\limits_{n\to\infty} n(a_n-1)=l\in(-\infty,\infty)$ and let $p\geq 1$ be a natural number. Calculate $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\left(a_n+\frac{1}{\sqrt[p]{kn}}\right)$.

Ovidiu Furdui

W14. 1). Let $f, g : [a,b]\to R$ be two nonnegative continuous functions. Assume that f attains its maximum at a unique point on $[a,b]$ and g attains its maximum at the same point as f and possibly at other points. Prove that $\lim\limits_{n\to\infty}\frac{\int_a^b f^{n+1}(x)g(x)dx}{\int_a^b f^n(x)dx}=\|f\|_\infty\|g\|_\infty$

2). Does the result hold under no assumption on f and g?

Ovidiu Furdui

W15. Let a, b, c be positive real numbers. Prove that

$$\sum_{\text{cyc}}\left(\frac{2a^2}{a+b}+\frac{a^3}{b^2+c^2}\right)\geq\frac{9}{2}\frac{a^2+b^2+c^2}{a+b+c}$$

Paolo Perfetti

W16. Prove that $\sin t\left(\cos^3 t+\sqrt{\sin t}\right)\geq\left(\frac{2t}{\pi}\right)^{\cos^2 t}$ when $0\leq t\leq\frac{\pi}{2}$.

Paolo Perfetti

W17. Let p a positive real number and let $\{a_n\}_{n\geq 1}$ be a sequence defined by $a_1=1, a_{n+1}=\frac{a_n}{1+a_n^p}$. Find those real values $q\neq 0$ such that the following series converges $\sum\limits_{n=1}^{\infty}\left|(pn)^{-\frac{1}{p}}-a_n\right|^q$.

Paolo Perfetti

W18. In 3-dimensional Euclidean space, let S be a sphere with centre O and radius R. Three pairwaise orthogonal rays originating in O intersect S at A, B, C. Let P be a point of S and let a, b, c denote the areas of triangles OPA, OPB, OPC, respectively. Prove $2a(+b+c)\left(a^3+b^3+c^3\right)\geq R^4(ab+bc+ca)$.

Michael Battaille

W19. Let a, b, c, x, y, z be real numbers such that $a, b, c > 0$ and $a + b + c = x + y + z = 1$. Let $u = xy - bz$, $v = az - cx$, $w = bx - ay$ and $M = \begin{pmatrix} \frac{2y}{b+c} & z - c & b - y \\ c - z & \frac{2v}{c+a} & x - a \\ y - b & a - x & \frac{2w}{a+b} \end{pmatrix}$. Prove that $\det(M) = 0$ if and only if $u + v + w = 0$.

Michael Battaille

W20. Let Γ be a circle and let $A \in \Gamma, B \in \Gamma, C \notin \Gamma$ be such that CA and CB intersect Γ again at diametrically opposite points. If l is the radial axis of Γ and the circle with centre A, radius AC, show that $d(C, l) = \frac{CA \cdot CB}{AB}$. ($d(C, l)$ denotes the distance from C to l).

Michael Battaille

W21. Let ABC be a triangle and we note $AB = c, BC = a, CA = b$ and m_a, m_b, m_c the medians lengths corresponding to the vertexes A, B respective C. Prove the inequality
$\sum \frac{\left(b^2+c^2\right)^2}{m_a^3} \geq \frac{32\sqrt{3}}{9}(a + b + c)$.

Ovidiu Pop

W22. Let $ABCD$ be a convex quadrilateral, $O \in [AC]$, $OM \parallel BC$, $M \in AB$, $ON \parallel AB$, $N \in BC$, $OP \parallel AD$, $P \in CD$ and $OQ \parallel CD$, $Q \in DA$. Prove the inequalities
$\min\{Area[ABD], Area[BCD]\} \leq Area[MNPQ] \leq$
$\leq \max\{Area[ABD], Area[BCD]\}$.

Ovidiu Pop

W23. Let $n \in N$, for k integer, $1 \leq k \leq n$, euclidean division n by k gives $n = qk + n_k$, and denote p_n the probability that $n_k \geq \frac{k}{2}$. Calculate p_n and find $\lim\limits_{n \to \infty} p_n$.

Moubinol Omarjee

W24. Let $f \in C^3(R^n, R)$ with $f(0) = f'(0) = 0$. Prove that there exist $h \in C^3(R^m, S_n(R))$, such that $f(x) = x^t h(x) x$, when $S_n(R)$, is the set of symmetric matrix, and x^t is the transpose of x.

Moubinol Omarjee

W25. Find the nature of the series $\sum\limits_{n \geq 1} \frac{e^{i \ln(p_n)}}{p_n}$ when $(p_n)_{n \geq 1}$ is the prime number increasing order, and i imaginary complex number.

Moubinol Omarjee

W26. Let M be a point in the interior of triangle ABC and R_a, R_b, R_c the radii of circumcircle of MBC, MCA, MAB. Show that $\frac{1}{R_a} + \frac{1}{R_b} + \frac{1}{R_c} \leq \frac{1}{MA} + \frac{1}{MB} + \frac{1}{MC}$

Nicuşor Minculete

W27. Let $a_j > 0$, $(j = 1, 2, \ldots, k)$ such that $\sum\limits_{cyclic} \prod\limits_{j=1}^{k-1} a_j = k$, and $n > 1$. Prove that

$$\sum_{cyclic} \sqrt[n]{a_1 + \frac{1}{\prod_{j=1}^{k} a_j}} \geq k\sqrt[n]{2}.$$

Ángel Plaza

W28. Let $(x_n)_{n\geq 0}$ be the sequence defined recurrently by $x_{n+2} = x_{n+1} - \frac{1}{2}x_n$ with initial terms $x_0 = 2$ and $x_1 = 1$. Find $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{n+2}$.

Ángel Plaza

W29. Let x, y, z be positive real numbers such that $x + y + z = 1$. Prove that
$$\sqrt{\frac{x^3+1}{x^2+y+z}} + \sqrt{\frac{y^3+1}{y^2+z+x}} + \sqrt{\frac{z^3+1}{z^2+x+y}} \leq 3\sqrt{2}.$$

Ángel Plaza

W30. Let be $x_0 = x_1 = 1$ and $x_{n+2} = 5x_{n+1} - x_n - 1$ for all $n \geq 0$. Prove
$x_n x_m x_p x_{m+1} x_{m+2} x_{p+2} \geq \left(\sqrt[3]{x_{n+1}^2 x_{m+1}} + \sqrt[3]{x_{m+1}^2 x_{p+1}} + \sqrt[3]{x_{p+1}^2 x_{n+1}}\right)^3$ for all $n, m, p \in N$.

Mihály Bencze

W31. If $a_k > 1$ $(k = 1, 2, ..., n)$ then $\sum \left(\log_{a_1} a_1 a_2\right)^{\lambda} \geq n2^{\lambda}$ for all $\lambda \in (-\infty, 0] \cup [1, +\infty)$.

Mihály Bencze

W32. If $x, y, z > 0$ and $x + y + z = 1$ then
$8(1-x)^x (1-y)^y (1-z)^z \leq 3(2-x-y)^{x+y} (2-y-z)^{y+z} (2-z-x)^{z+x}$.

Mihály Bencze

W33. 1). Let $f : R \to R$ be a bijective and continuous function such that $f(a) = \lambda a$ and $f(b) = \lambda b$ when $a, b \in R$ and $\lambda \in R^*$. Prove that exist $x_0 \in (a, b)$ such that $f(x_0) + \lambda f^{-1}(\lambda x_0) = 2\lambda x_0$.
2). Let $f : R \to R$ be a bijective and continuous function such that $f(a) = \lambda b$ and $f(b) = \lambda a$ when $a, b \in R$ and $\lambda \in R^*$. Prove that exist $x_0 \in (a, b)$ such that $f\left(\frac{f(x_0)}{\lambda}\right) = \lambda x_0$.

Mihály Bencze

W34. Prove $\sum\limits_{k=1}^{n} \left(\frac{2(k+1)(k+2)^2}{((k+2)!)^3}\right)^{\frac{1}{k}} \geq \frac{n(n+5)}{3(n+2)(n+3)}$.

Mihály Bencze

W35. Prove that exist infinitely many $n \in N$ for which $n!$ is divisible by $n^5 + n - 1$.

Mihály Bencze

W36. Let $\triangle(x,y,z) = 2(xy+yz+zx) - \left(x^2+y^2+z^2\right)$ and let a,b,c be sidelengths of a triangle with area F. Prove that $\triangle\left(a^3,b^3,c^3\right) \le \frac{64F^3}{\sqrt{3}}$.

Arkady Alt

W37. Let E be a inner Product Space with dot product $- \cdot -$ and F be proper nonzero subspace. Let $P: E \to E$ be orthogonal proiection E on F.
a). Prove that for any $x,y \in E$, holds inequality $|x \cdot y - xP(y) - yP(x)| \le \|x\| \cdot \|y\|$
b). Determine all cases when equality occours

Arkady Alt

W38. Prove that $0 < \left(\frac{4^x+2^x+1}{x}\right)^x - 2^x < 1$ for all $x \in \left(0, \frac{1}{2e}\right]$.

Ionel Tudor

W39. Let $n \ge 2$ be a natural number and $a_i > 0$, $i = \overline{1,n}$. If $S = \sum_{i=1}^n a_i$ and $x_i = S - a_i$, then the following inequality holds:

$$\frac{\prod_{i=1}^n \sqrt{a_i}}{\sqrt[(n-1)]{\prod_{1\le i<j\le n}(a_i+a_j)}} \le \frac{\prod_{i=1}^n \sqrt{x_i}}{\sqrt[(n-1)]{\prod_{1\le i<j\le n}(x_i+x_j)}}.$$

Ovidiu Bagdasar

W40. Prove that if $x_i > 0$, $i = \overline{1,n}$, then the next inequality holds:

$$\sum_{i=1}^n \frac{S_{\alpha+\beta} - x_i^{\alpha+\beta}}{S_\alpha - x_i^\alpha} \le n \cdot \frac{S_{\alpha+\beta}}{S_\alpha}, \tag{1}$$

provided that $\alpha\beta \ge 0$ and $S_p = \sum_{i=1}^n x_i^p$, for any real number p.

Ovidiu Bagdasar

W41. Let $n \ge 2$ a natural number and the numbers $a_i > 1$, $i = \overline{1,n}$. Prove that

$$\sum_{i=1}^n \frac{\log_{a_i} a_{i+1}^{n-1}}{S - a_i} \ge \frac{n^2}{\sum_{i=1}^n a_i}.$$

We consider that $a_{n+1} = a_1$, and $S = \sum\limits_{i=1}^n a_i$.

Ovidiu Bagdasar

W42. Let ABC be an acute triangle. The angle bisectors from A, B, C meet the opposite sides in A_1, B_1, C_1, respectively. Let R and r be the circumradius and the inradius of the triangle ABC, respectively. Let R_A, R_B, and R_C the circumradii of the triangles AC_1B_1, BA_1C_1, and CB_1A_1, respectively. Prove that

$$R_A + R_B + R_C \ge R + r.$$

Pál Péter Dályay

W43. Let f be a continuous real function defined on the set of the nonnegative real numbers for which the following integrals are convergent: $S = \int_0^\infty f^2(x)dx$, $T = \int_0^\infty xf^2(x)dx$, $U = \int_0^\infty x^2f^2(x)dx$. Prove that

$$\left(\int_0^\infty |f(x)|dx\right)^2 \le 2\left(\sqrt{SU}+T\right).$$

Pál Péter Dályay

W44. If ζ is the Riemann zeta-function, and s is a real number greater than $3/2$, then:

$$\zeta^2(s) \le \pi\sqrt{2}\left(\sum_{i,j=1}^{\infty}\frac{1}{i^{s-1}j^{s-1}(i+j)}\sum_{k,l=1}^{\infty}\frac{1}{k^{s-1}l^{s-1}(k+l)^3}\right)^{1/2} \le$$
$$\le \frac{\pi}{2\sqrt{2}}\zeta(s-1/2)\zeta(s+1/2).$$

Pál Péter Dályay

W45. Let $x \in Poisson\,(2)$ be a random variable
i). Find the set M of all the values $n \in N^*$ so that $P\left(\left\{\omega/\left|x\left(\omega\right)-2\right| \ge \frac{2}{n}\right\}\right) \le \frac{128}{n^2}$
ii). From the set M we extract 2 numbers one, after the other. Find the probability that the second extradet number could be the greatest, in the following two situation: with return in the set, and without return in M.

Laurenţiu Modan

W46. i). For any $n > 3$ natural numbers, prove that $2n! + (2n)! > 3 \cdot 2^{n+2}$
ii). Study the convergence of the series $\sum\limits_{n>3}\frac{1}{2n!+(2n)!}$

Laurenţiu Modan

W47. Let $a, b, c, d \in (0,1)$ and $f : (0,1) \to R$ a convex and decreasing function. Prove that
$\sum f\left(1-a^3\right) \ge \sum f\left(1-a^2b\right).$

Marius Drăgan

W48. Let I be an interval and $f : I \to R$ a convex function and $x_1, x_2, ..., x_n \in I$. Prove that
$\sum\limits_{k=1}^{n} f\left(x_k\right) - n\left(\frac{1}{n}\sum\limits_{k=1}^{n} x_k\right) \ge \max\limits_{1\le i<...<i_k\le n}\left(f\left(x_{i_1}\right)+...+f\left(x_{i_k}\right)-kf\left(\frac{x_{i_1}+...+x_{i_k}}{k}\right)\right)$

Marius Drăgan and Mihály Bencze

W49. Find all the functions $f : R \to [-1,1]$ such that $f\left(x+y\right)\left(1-f\left(x\right)f\left(y\right)\right) = f\left(x\right)+f\left(y\right)$ for each $x, y \in R$.

Marius Drăgan and Sorin Rădulescu

W50. Find all the function $f : R \to R$ which are continuous in a real point x_0 such that
$f\left(x+y\right) = f\left(x\right)\sqrt{1+f^2\left(y\right)} + f\left(y\right)\sqrt{1+f^2\left(x\right)}$ for each $x, y \in R$.

Sorin Rădulescu