Test 1 — Solutions

Problem 1. Two circles, ω_1 and ω_2, centred at O_1 and O_2, respectively, meet at points A and B. A line through B meets ω_1 again at C, and ω_2 again at D. The tangents to ω_1 and ω_2 at C and D, respectively, meet at E, and the line AE meets the circle ω through A, O_1, O_2 again at F. Prove that the length of the segment EF is equal to the diameter of ω.

Solution. Begin by noticing that the lines CO_1 and DO_2 meet at a point P on ω, since $\angle(PO_1, PO_2) = \angle(O_1C, CB) + \angle(BD, DO_2) = \angle(CB, BO_1) + \angle(O_2B, BD) = \angle(O_2B, BO_1) = \angle(O_1A, AO_2)$. In what follows, we consider the case where O_1 and O_2 lie on the segments CP and DP, respectively; the other cases are similar.

Since the angles PCE and PDE are both right, and $2\angle ACP = \angle AO_1P = \angle AO_2P = 2\angle ADP$ (the equality in the middle holds on account of P lying on ω), the points A, C, D, E, P all lie on the circle on diameter EP, so FP is a diameter of ω, and it is therefore sufficient to show that $EF = FP$. Finally, since $\angle AFP = \angle AO_1P = 2\angle ACP = 2\angle AEP$ (the first, respectively third, equality holds on account of $APFO_1$, respectively $ACEP$, being cyclic), it follows that the triangle EFP is isosceles with apex at F.

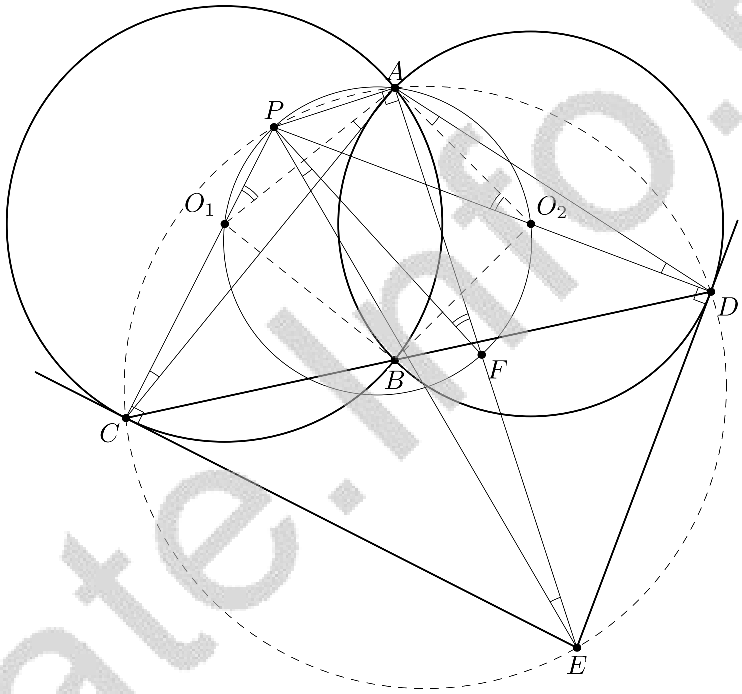

Fig. 1

Problem 2. Let n be a positive integer, and let $S_1, \ldots, S_n$ be a collection of finite non-empty sets such that

$$\sum_{1\le i<j\le n} \frac{|S_i \cap S_j|}{|S_i|\,|S_j|} < 1.$$

Prove that there exist pairwise distinct elements $x_1, \ldots, x_n$ such that x_i is a member of S_i for each index i.

Solution. A *choice function* or simply a *choice* for the collection $S_1, \ldots, S_n$ is a function c from the first n positive integers to the union $S_1 \cup \ldots \cup S_n$ such that $c(i)$ is a member of S_i for each i. We must show that an injective choice is always possible under the conditions in the statement. To this end, we prove that the number of non-injective choices is strictly less than $|S_1| \cdots |S_n|$, the total number of possible choices. Indeed, a non-injective choice function sends some i and some $j \neq i$ to a same element necessarily lying in $S_i \cap S_j$, so the number of non-injective choices does not exceed

$$\sum_{1 \le i < j \le n} |S_i \cap S_j|\,|S_1| \cdots |\hat{S}_i| \cdots |\hat{S}_j| \cdots |S_n| = |S_1| \cdots |S_n| \sum_{1 \le i < j \le n} \frac{|S_i \cap S_j|}{|S_i|\,|S_j|} < |S_1| \cdots |S_n|;$$

the hat over S_i and S_j means that these sets are to be omitted. The conclusion follows.

Problem 3. Let n be a positive integer, and let $a_1, \ldots, a_n$ be pairwise distinct positive integers. Show that

$$\sum_{k=1}^{n} \frac{1}{[a_1, \ldots, a_k]} < 4,$$

where $[a_1, \ldots, a_k]$ is the least common multiple of the integers $a_1, \ldots, a_k$.

Solution. Since the number of positive divisors of a positive integer m does not exceed $2\sqrt{m}$, and $a_1, \ldots, a_k$ are pairwise distinct positive divisors of $[a_1, \ldots, a_k]$, it follows that $[a_1, \ldots, a_k] \ge k^2/4$. Consequently,

$$\begin{aligned} \sum_{k=1}^{n} \frac{1}{[a_1, \ldots, a_k]} &= \frac{1}{a_1} + \sum_{k=2}^{n} \frac{1}{[a_1, \ldots, a_k]} \le 1 + \sum_{k=2}^{n} \frac{4}{k^2} < 1 + 4 \sum_{k=2}^{n} \frac{1}{k^2 - \frac{1}{4}} \\ &= 1 + 4 \cdot 2 \left(\frac{1}{3} - \frac{1}{2n+1} \right) < \frac{11}{3} < 4. \end{aligned}$$

Problem 4. Determine the integers $k \ge 2$ for which the sequence $\binom{2n}{n} \pmod{k}$, $n = 0, 1, 2, \ldots$, is eventually periodic.

Solution. Since $\binom{2n}{n} = 2\binom{2n-1}{n} \equiv 0 \pmod{2}$, $n = 1, 2, 3, \ldots$, it follows that 2 satisfies the required condition. We will prove that no $k \ge 3$ does.

If d is a divisor of an integer k, and the sequence $\binom{2n}{n} \pmod{k}$, $n = 0, 1, 2, \ldots$, is eventually periodic, then so is the sequence $\binom{2n}{n} \pmod{d}$, $n = 0, 1, 2, \ldots$.

We will show that every integer $k \ge 3$ has a divisor d such that the sequence $\binom{2n}{n} \pmod{d}$, $n = 0, 1, 2, \ldots$, has arbitrarily long stretches of consecutive 0's and non-zero terms of arbitrarily large rank. It then follows that this sequence is not eventually periodic, so the sequence $\binom{2n}{n} \pmod{k}$, $n = 0, 1, 2, \ldots$, is not eventually periodic either.

To prove that such a divisor exists, notice that an integer $k \ge 3$ is either divisible by 4 or else has at least one odd prime divisor p. We will prove that $d = 4$ works in the former case, and $d = p$ does in the latter.

If k is a positive integer divisible by 4, consider a positive integer m, and let $n = 2^m + r$, $r = 0, 1, \ldots, 2^m - 1$, to write

$$(1+X)^{2n} = (1+X)^{2^{m+1}} (1+X)^{2r} \equiv_4 \left(1 + 2X^{2^m} + X^{2^{m+1}}\right) (1+X)^{2r},$$

and infer that

$$\binom{2n}{n} \equiv_4 2\binom{2r}{r} \equiv_4 \begin{cases} 2 & \text{if } r = 0, \\ 4\binom{2r-1}{r} \equiv_4 0 & \text{if } r = 1, \ldots, 2^m - 1. \end{cases}$$

Since m is arbitrary, the sequence $\binom{2n}{n} \pmod{4}$, $n = 0, 1, 2, \ldots$, has arbitrarily long stretches of consecutive 0's and non-zero terms of arbitrarily large rank.

If k is a positive integer divisible by an odd prime p, consider again a positive integer m, let $n = (p^m + r)/2$, where r runs through the positive odd integers not exceeding p^m, to write

$$\begin{aligned} (1+X)^{2n} &= (1+X)^{p^m} (1+X)^r \equiv_p \left(1 + X^{p^m}\right) (1+X)^r \\ &= \text{terms of degree} < r + X^r + X^{p^m} + \text{terms of degree} > p^m, \end{aligned}$$

and infer that $\binom{2n}{n} \equiv_p 0$ if r is less than p^m, since in this case $r < n < p^m$, and $\binom{2n}{n} \equiv_p 2$ if $r = p^m = n$. Since m is arbitrary, the sequence $\binom{2n}{n} \pmod{p}$, $n = 0, 1, 2, \ldots$, has arbitrarily long stretches of consecutive 0's and non-zero terms of arbitrarily large rank.