COMBINATORICA

1. PERMUTARI

Fie E={1, 2, …,n} o multime finita cu n elemente. Se numeste permutare a multimii E orice functie bijectiva $f : E \to E$.

Notam permutarea in felul urmator

$$f : \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ f(1) & f(2) & f(3) & \dots & f(n) \end{pmatrix}$$

Notam numarul de permutari Pn: **$P_n = n! = 1.2.3 \dots n$**

conditie de existenta: $n \in \mathbb{N}$

conventie: $0!=1$; $1!=1$

$P_n = n(n-1)! = n(n-1)(n-2)!$

2. ARANJAMENTE

Notam cu A_n^k

Sistemele ordonate cu k elemente, care se pot forma cu elementele unei multimi cu n elemente ($n \geq k$), se numesc aranjamente de n elemente luate cate k.

$A_n^k = n!/(n-k)! = n(n-1)(n-2)\dots(n-k+1) = (n-k+1)A_n^{k-1}$

c.e. $n \geq k$

conventie: $n=k \Rightarrow A_n^n = P_n$

3. COMBINARI C_n^k

$$C_n^k = \frac{A_n^k}{P_n} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)\dots(n-k+1)}{1\cdot 2\cdot 3\cdot 4\dots k} = \frac{n-k+1}{k} C_n^{k-1}$$

conventie: $C_n^0 = C_n^n = 1$ c.e. $n \geq k$

Formule pentru combinari complementare: $C_n^k = C_n^{n-k}$

$C_n^k = C_{n-1}^k + C_{n-1}^{k-1}$

4. BINOMUL LUI NEWTON

Daca $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, atunci:

$(a+b)^n = C_n^0 a^n + C_n^1 a^{n-1} b + C_n^2 a^{n-2} b^2 + \dots + C_n^k a^{n-k} b^k + \dots + C_n^{n-1} a b^{n-1} + C_n^n b^n$

sau

$$(a+b)^n = \sum_{k=0}^{n} C_n^k a^{n-k} b^k = \sum_{k=0}^{n} T_{k+1}, \text{ unde } T_{k+1} = C_n^k a^{n-k} b^k$$

T_{k+1}=termen general

k=se numeste rangul termenului al dezvoltarii

$$(a-b)^n = C_n^0a^n - C_n^1a^{n-1}b + C_n^2a^{n-2}b^2 - \ldots + (-1)^{n-k}C_n^ka^{n-k}b^k + \ldots + (-1)^{n-1}C_n^{n-1}ab^{n-1} + (-1)^nC_n^nb^n$$

sau

$$(a-b)^n = (-1)^k \sum_{k=0}^{n} C_n^k a^{n-k} b^k = \sum_{k=0}^{n} T_{k+1}, \text{ unde } T_{k+1} = (-1)^n C_n^k a^{n-k} b^k$$

Obs: 1) in dezvoltarea $(a+b)^n$, dupa formula lui Newton, sunt n+1 termeni.

2) $C_n^0, C_n^1, C_n^2, \ldots, C_n^n$ se numesc coeficienti binomiali

3) Sa se faca distinctie intre coeficientul unui termen al dezvoltarii si coeficientul binomial al aceluiasi termen.

4) Pentru a determina rangul celui mai mare termen folosim relatia:

$$\frac{T_{k+2}}{T_{k+1}} = \frac{n-k}{k+1} \cdot \frac{b}{a} \quad \text{si studiem doua cazuri:}$$

$$\frac{T_{k+2}}{T_{k+1}} \geq 1 \qquad \text{si} \qquad \frac{T_{k+2}}{T_{k+1}} < 1$$

5) In dezvoltarea $(a+b)^n$ si $(a-b)^n$, daca a=b atunci:

$$C_n^0 + C_n^1 + C_n^2 + \ldots + C_n^n = 2^n$$

$$C_n^0 + C_n^2 + C_n^4 + \ldots = C_n^1 + C_n^3 + C_n^5 + \ldots = 2^{n-1}$$

6) Identitatile utile:

a) $C_n^k = C_{n-1}^{k-1} + C_{n-2}^{k-1} + \ldots + C_{k-1}^{k-1}$

b) $C_{n+k}^k = C_n^0C_m^k + C_n^1C_m^{k-1} + \ldots + C_n^kC_m^0$

7) Suma puterilor asemenea ale primelor n numere naturale

Fie $k \geq 1$ un numar natural si $S_k = 1^k + 2^k + 3^k + \ldots + n^k$

a) $S_1 = 1+2+3+\ldots+n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Folosim dezvoltarea $(a+1)^2 = a^2 + 2a + 1$ pentru demonstratie unde a=1,2,…n.

b) $S_2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Folosim dezvoltarea $(a+1)^3 = a^3 + 3a^2 + 3a + 1$, pentru demonstratie, unde a=1,2,…n.
Folosim dezvoltarea $(a+1)^4 = a^4 + 4a^3 + 6a^2 + 4a + 1$, pentru demonstratie, unde

c) $S_3 = 1^3 + 2^3 + 3^3 + \ldots n^3 = \sum_{k=1}^{n} k^3 = [\frac{n(n+1)}{2}]^2$

d) $S_k=\dfrac{(n+1)^{k+1}-(n+1)-\sum_{p=2}^{k} C_{k+1}^{p}\cdot S_{p-1}}{C_{k+1}^{1}}$

a=1,2,...n

Caz particular

$$S_4=\frac{n(n+1)(2n+1)(3n2+3n-1)}{30}$$

5. PROGRESII ARITMETICE SI GEOMETRICE

a) PROGRESII ARITMETIVE ÷

Teorema : Fie numerele a_{n-1}, a_n, a_{n+1} in progresie aritmetica. Atunci:

$$2a_n=a_{n-1}+a_{n+1}$$

Def: Fie numerele a_1, a_2, a_3,...,a_n in progresie aritmetica, daca $a_n=a_1+(n-1)r$ sau $a_n=a_{n-1}+1$, unde: a_n= ultimul termen

a_1=primul termen

a_{n-1}=penultimul termen

n=numarul de termeni

r=ratia progresiei aritmetice

Obs: Pentru verificare $r=a_2-a_1=a_3-a_2=a_4-a_3=...=a_n-a_{n-1}$

$$\text{P:}\quad Sn=a_1+a_2+...+a_n=\frac{(a_1+a_n)n}{2}=\frac{[2a_1+(n-1)r]n}{2}$$

b) PROGRESII GEOMETRICE ÷÷

Teorema: Fie numerele b_{n-1}, b_n, b_{n+1} in progresie geometrica. Atunci

$$b_n^2=b_{n-1}\cdot b_{n+1}$$

Def: Fie numerele b_1, b_2,...b_n in progresie geometrica, daca $b_n=b_1\cdot q^n$ sau $b_n=b_{n-1}\cdot q$ unde: b_n=ultimul termen

b_1=primul termen

b_{n-1}=penultimul termen

n=numarul de termeni

q=ratia progresiei geometrice

Obs: pentru verificare $q=a_2/a_1=a_3/a_2=...=a_n/a_{n-1}$

$$\text{P:}\ Sn=b_1(1+q+q^2+q^3+...+q^{n-1})=b_1\cdot\frac{q^n-1}{q-1}=b_1+b_2+...+b_n$$