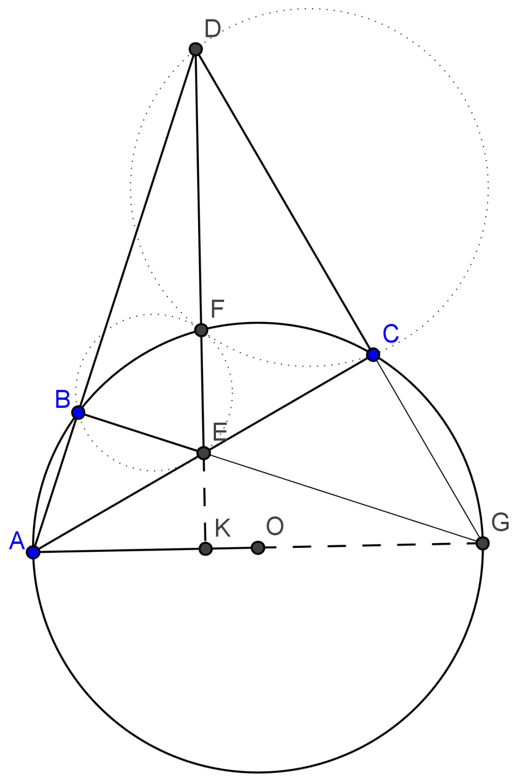
Problem 1.

Solution. Let $\ell \cap AO = \{K\}$ and G be the other end point of the diameter of Γ through A. Then D, C, G are collinear. Moreover, E is the orthocenter of triangle ADG. Therefore $GE \perp AD$ and G, E, B are collinear.



As $\angle CDF = \angle GDK = \angle GAC = \angle GFC$, FG is tangent to the circumcircle of triangle CFD at F. As $\angle FBE = \angle FBG = \angle FAG = \angle GFK = \angle GFE$, FG is also tangent to the circumcircle of BFE at F. Hence the circumcircles of the triangles CFD and BFE are tangent at F.

Problem 2.

Solution 1. We will obtain the inequality by adding the inequalities

$$(x+y)\sqrt{(z+x)(z+y)} \geq 2xy + yz + zx$$

for cyclic permutation of x, y, z.

Squaring both sides of this inequality we obtain

$$(x+y)^2(z+x)(z+y) \geq 4x^2y^2 + y^2z^2 + z^2x^2 + 4xy^2z + 4x^2yz + 2xyz^2$$

which is equivalent to

$$x^3y + xy^3 + z(x^3+y^3) \geq 2x^2y^2 + xyz(x+y)$$

which can be rearranged to

$$(xy+yz+zx)(x-y)^2 \geq 0,$$

which is clearly true.

Solution 2. For positive real numbers x, y, z there exists a triangle with the side lengths $\sqrt{x+y}, \sqrt{y+z}, \sqrt{z+x}$ and the area $K = \sqrt{xy+yz+zx}/2$.

The existence of the triangle is clear by simple checking of the triangle inequality. To prove the area formula, we have

$$K = \frac{1}{2}\sqrt{x+y}\sqrt{z+x}\sin\alpha,$$

where α is the angle between the sides of length $\sqrt{x+y}$ and $\sqrt{z+x}$. On the other hand, from the law of cosines we have

$$\cos\alpha = \frac{x+y+z+x-y-z}{2\sqrt{(x+y)(z+x)}} = \frac{x}{\sqrt{(x+y)(z+x)}}$$

and

$$\sin\alpha = \sqrt{1-\cos^2\alpha} = \frac{\sqrt{xy+yz+zx}}{\sqrt{(x+y)(z+x)}}.$$

Now the inequality is equivalent to

$$\sqrt{x+y}\sqrt{y+z}\sqrt{z+x}\sum_{cyc}\sqrt{x+y} \geq 16K^2.$$

This can be rewritten as

$$\frac{\sqrt{x+y}\sqrt{y+z}\sqrt{z+x}}{4K} \geq 2\,\frac{K}{\sum_{cyc}\sqrt{x+y}/2}$$

to become the Euler inequality $R \geq 2r$.

Problem 3.

Solution 1. Let $\alpha = 3/2$ so $1 + \alpha > \alpha^2$.

Given y, we construct Y algorithmically. Let $Y = \varnothing$ and of course $S_\varnothing = 0$. For $i = 0$ to m, perform the following operation:

$$\text{If } S_Y + 2^i 3^{m-i} \leq y, \text{ then replace } Y \text{ by } Y \cup \{2^i 3^{m-i}\}.$$

When this process is finished, we have a subset Y of P_m such that $S_Y \leq y$.

Notice that the elements of P_m are in ascending order of size as given, and may alternatively be described as $2^m, 2^m\alpha, 2^m\alpha^2, \ldots, 2^m\alpha^m$. If any member of this list is not in Y, then no two consecutive members of the list to the left of the omitted member can both be in Y. This is because $1 + \alpha > \alpha^2$, and the greedy nature of the process used to construct Y.

Therefore either $Y = P_m$, in which case $y = 3^{m+1} - 2^{m+1}$ and all is well, or at least one of the two leftmost elements of the list is omitted from Y.

If 2^m is not omitted from Y, then the algorithmic process ensures that $(S_Y - 2^m) + 2^{m-1}3 > y$, and so $y - S_Y < 2^m$. On the other hand, if 2^m is omitted from Y, then $y - S_Y < 2^m$).

Solution 2. Note that $3^{m+1} - 2^{m+1} = (3-2)(3^m + 3^{m-1} \cdot 2 + \cdots + 3 \cdot 2^{m-1} + 2^m) = S_{P_m}$. Dividing every element of P_m by 2^m gives us the following equivalent problem:

Let m be a positive integer, $a = 3/2$, and $Q_m = \{1, a, a^2, \ldots, a^m\}$. Show that for any real number x satisfying $0 \leq x \leq 1 + a + a^2 + \cdots + a^m$, there exists a subset X of Q_m such that $0 \leq x - S_X < 1$.

We will prove this problem by induction on m. When $m = 1$, $S_\varnothing = 0$, $S_{\{1\}} = 1$, $S_{\{a\}} = 3/2$, $S_{\{1,a\}} = 5/2$. Since the difference between any two consecutive of them is at most 1, the claim is true.

Suppose that the statement is true for positive integer m. Let x be a real number with $0 \leq x \leq 1 + a + a^2 + \cdots + a^{m+1}$. If $0 \leq x \leq 1 + a + a^2 + \cdots + a^m$, then by the induction hypothesis there exists a subset X of $Q_m \subset Q_{m+1}$ such that $0 \leq x - S_X < 1$.

If $\dfrac{a^{m+1} - 1}{a - 1} = 1 + a + a^2 + \cdots + a^m < x$, then $x > a^{m+1}$ as

$$\frac{a^{m+1} - 1}{a - 1} = 2(a^{m+1} - 1) = a^{m+1} + (a^{m+1} - 2) \geq a^{m+1} + a^2 - 2 = a^{m+1} + \frac{1}{4}.$$

Therefore $0 < (x - a^{m+1}) \leq 1 + a + a^2 + \cdots + a^m$. Again by the induction hypothesis, there exists a subset X of Q_m satisfying $0 \leq (x - a^{m+1}) - S_X < 1$. Hence $0 \leq x - S_{X'} < 1$ where $X' = X \cup \{a^{m+1}\} \subset Q_{m+1}$.

Problem 4.

Solution 1. There are three such functions: the constant functions 1, 2 and the identity function $\mathrm{id}_{\mathbf{Z}^+}$. These functions clearly satisfy the conditions in the hypothesis. Let us prove that there are only ones.

Consider such a function f and suppose that it has a fixed point $a \geq 3$, that is $f(a) = a$. Then $a!, (a!)!, \cdots$ are all fixed points of f, hence the function f has a strictly increasing sequence $a_1 < a_2 < \cdots < a_k < \cdots$ of fixed points. For a positive integer n, $a_k - n$ divides $a_k - f(n) = f(a_k) - f(n)$ for every $k \in \mathbf{Z}^+$. Also $a_k - n$ divides $a_k - n$, so it divides $a_k - f(n) - (a_k - n) = n - f(n)$. This is possible only if $f(n) = n$, hence in this case we get $f = \mathrm{id}_{\mathbf{Z}^+}$.

Now suppose that f has no fixed points greater than 2. Let $p \geq 5$ be a prime and notice that by Wilson's Theorem we have $(p-2)! \equiv 1 \pmod{p}$. Therefore p divides $(p-2)! - 1$. But $(p-2)! - 1$ divides $f((p-2)!) - f(1)$, hence p divides $f((p-2)!) - f(1) = (f(p-2))! - f(1)$. Clearly we have $f(1) = 1$ or $f(1) = 2$. As $p \geq 5$, the fact that p divides $(f(p-2))! - f(1)$ implies that $f(p-2) < p$. It is easy to check, again by Wilson's Theorem, that p does not divide $(p-1)! - 1$ and $(p-1)! - 2$, hence we deduce that $f(p-2) \leq p-2$. On the other hand, $p - 3 = (p-2) - 1$ divides $f(p-2) - f(1) \leq (p-2) - 1$. Thus either $f(p-2) = f(1)$ or $f(p-2) = p-2$. As $p - 2 \geq 3$, the last case is excluded, since the function f has no fixed points greater than 2. It follows $f(p-2) = f(1)$ and this property holds for all primes $p \geq 5$. Taking n any positive integer, we deduce that $p - 2 - n$ divides $f(p-2) - f(n) = f(1) - f(n)$ for all primes $p \geq 5$. Thus $f(n) = f(1)$, hence f is the constant function 1 or 2.

Solution 2. Note first that if $f(n_0) = n_0$, then $m - n_0 | f(m) - m$ for all $m \in \mathbf{Z}^+$. If $f(n_0) = n_0$ for infinitely many $n_0 \in \mathbf{Z}^+$, then $f(m) - m$ has infinitely many divisors, hence $f(m) = m$ for all $m \in \mathbf{Z}^+$. On the other hand, if $f(n_0) = n_0$ for some $n_0 \geq 3$, then f fixes each term of the sequence $(n_k)_{k=0}^{\infty}$, which is recursively defined by $n_k = n_{k-1}!$. Hence if $f(3) = 3$, then $f(n) = n$ for all $n \in \mathbf{Z}^+$.

We may assume that $f(3) \neq 3$. Since $f(1) = f(1)!$, and $f(2) = f(2)!$, $f(1), f(2) \in \{1, 2\}$. We have $4 = 3! - 2 | f(3)! - f(2)$. This together with $f(3) \neq 3$ implies that $f(3) \in \{1, 2\}$. Let $n > 3$, then $n! - 3 | f(n)! - f(3)$ and $3 \nmid f(n)!$, i.e. $f(n)! \in \{1, 2\}$. Hence we conclude that $f(n) \in \{1, 2\}$ for all $n \in \mathbf{Z}^+$.

If f is not constant, then there exist positive integers m, n with $\{f(n), f(m)\} = \{1, 2\}$. Let $k = 2 + \max\{m, n\}$. If $f(k) \neq f(m)$, then $k - m | f(k) - f(m)$. This is a contradiction as $|f(k) - f(m)| = 1$ and $k - m \geq 2$.

Therefore the functions satisfying the conditions are $f \equiv 1, f \equiv 2, f = \mathrm{id}_{\mathbf{Z}^+}$.