Subiecte pentru pregătirea Olimpiadei de matematică – clasa a VIII-a

1. a) Ştiind că : $x^3 - 1 = y(x-1)$ şi $x \neq 1$ pentru $(\forall)$ x, y $\in \mathbf{R}$,
arătaţi că $y > 0$

(2puncte)

b) Aflaţi valoarea reală a numărului

$$\mathbf{E} = x^4 + x^3 + x^2 + x + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} \quad \textbf{ştiind că} \quad \left(x - \frac{1}{x}\right)^3 = 64$$

(5puncte)

Soluţie

a) $x^3 - 1 = (x-1)(x^2 + x + 1) \Rightarrow y = x^2 + x + 1$

$y = x^2 + x + 1 = x^2 + 2 \cdot \frac{1}{2} x + \frac{1}{4} + \frac{3}{4} = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

dar $\left(x + \frac{1}{2}\right)^2 \geq 0$, $\forall$ $x \in \mathbf{R}$ $\Rightarrow$ $\left(x + \frac{1}{2}\right)^2 + \frac{3}{4} > 0 \Rightarrow y > 0$

b) $\left(x - \frac{1}{x}\right)^3 = 64 \quad \Rightarrow \quad x - \frac{1}{x} = 4 \quad \Rightarrow \left(x - \frac{1}{x}\right)^2 = 16$

$\left(x - \frac{1}{x}\right)^2 = 16 \quad \Rightarrow x^2 + \frac{1}{x^2} = 18 \quad \Rightarrow x^2 + \frac{1}{x^2} + 2 = 20 \quad \Rightarrow \left(x + \frac{1}{x}\right)^2 = 20$

$\Rightarrow \left|x + \frac{1}{x}\right| = \sqrt{20} \quad \Rightarrow \quad x + \frac{1}{x} = 2\sqrt{5}$ **sau** $x + \frac{1}{x} = -2\sqrt{5}$

Cazul I.

$x + \frac{1}{x} = 2\sqrt{5} \quad \Rightarrow \quad \left(x + \frac{1}{x}\right)^3 = 40\sqrt{5} \quad \Rightarrow \quad x^3 + \frac{1}{x^3} = 34\sqrt{5}$

$\left(x^2 + \frac{1}{x^2}\right)^2 = 18 \quad \Rightarrow \quad x^4 + \frac{1}{x^4} = 322$

$E_1 = \left(x^4 + \frac{1}{x^4}\right) + \left(x^3 + \frac{1}{x^3}\right) + \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) = 322 + 34\sqrt{5} + 18 + 2\sqrt{5} = 340 + 36\sqrt{5}$

Cazul II. $x + \frac{1}{x} = -2\sqrt{5} \quad \Rightarrow \quad \left(x + \frac{1}{x}\right)^3 = -40\sqrt{5} \quad \Rightarrow \quad x^3 + \frac{1}{x^3} = -34\sqrt{5}$

$x^2 + \frac{1}{x^2} = 18$ şi $x^4 + \frac{1}{x^4} = 322$

$E_2 = \left(x^4 + \frac{1}{x^4}\right) + \left(x^3 + \frac{1}{x^3}\right) + \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) = 322 - 34\sqrt{5} + 18 - 2\sqrt{5} = 340 - 36\sqrt{5}$

2. Arătați că numărul a = $\frac{3\cdot5\cdot7\cdot9\cdot\ldots\cdot2009}{4\cdot6\cdot8\cdot10\cdot\ldots\cdot2010}$ aparține intervalului $\left(\frac{\sqrt{2}}{45};\frac{\sqrt{3}}{44}\right)$

(7 puncte)

<u>Soluție</u>

$$a\in\left(\frac{\sqrt{2}}{45};\frac{\sqrt{3}}{44}\right) \Rightarrow \frac{\sqrt{2}}{45}< a <\frac{\sqrt{3}}{44} \Rightarrow \frac{2}{2025}< a^2 <\frac{3}{1936}$$

$$a^2=\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}\cdot\ldots\cdot\frac{2009^2}{2010^2}>\frac{3^2-1}{4^2}\cdot\frac{5^2-1}{6^2}\cdot\frac{7^2-1}{8^2}\cdot\ldots\cdot\frac{2009^2-1}{2010^2}=$$

$$=\frac{2\cdot4}{4^2}\cdot\frac{4\cdot6}{6^2}\cdot\frac{6\cdot8}{8^2}\cdot\ldots\cdot\frac{2008\cdot2010}{2010^2}=\frac{2}{4}\cdot\frac{4}{6}\cdot\frac{6}{8}\cdot\ldots\cdot\frac{2008}{2010}=\frac{2}{2010}>\frac{2}{2025}$$

$$a^2=\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\cdot\frac{7^2}{8^2}\cdot\ldots\cdot\frac{2009^2}{2010^2}<\frac{3^2}{4^2-1}\cdot\frac{5^2}{6^2-1}\cdot\frac{7^2}{8^2-1}\cdot\ldots\cdot\frac{2009^2}{2010^2-1}=$$

$$=\frac{3^2}{3\cdot5}\cdot\frac{5^2}{5\cdot7}\cdot\frac{7^2}{7\cdot9}\cdot\ldots\cdot\frac{2009^2}{2009\cdot2011}=\frac{3}{5}\cdot\frac{5}{7}\cdot\frac{7}{9}\cdot\ldots\cdot\frac{2009}{2011}=\frac{3}{2011}<\frac{3}{1936}$$

$$\begin{cases} a^2>\dfrac{2}{2025} \\ a^2<\dfrac{3}{1936}\end{cases} \Rightarrow \begin{array}{l}\dfrac{2}{2025}<a^2<\dfrac{3}{1936} \\ a>0\end{array} \Rightarrow \frac{\sqrt{2}}{45}< a <\frac{\sqrt{3}}{44} \Rightarrow$$

$$a\in\left(\frac{\sqrt{2}}{45};\frac{\sqrt{3}}{44}\right)$$

3. Fie x_1 , x_2 , x_3 ,...... , x_n numere reale . Arătați că dacă

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 + \ldots . X_n^2 = \frac{(x_1 + x_2 + x_3 + \ldots . x_n)^2}{n},$$

atunci $x_1 = x_2 = x_3 = \ldots\ldots = x_n$

(4 puncte)

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Soluție

$$\frac{x_1 + x_2 + x_3 + \ldots . x_n}{n} = a \Rightarrow X_1^2 + X_2^2 + X_3^2 + X_4^2 + \ldots . X_n^2 = n \cdot a^2$$

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 + \ldots . X_n^2 - 2n \cdot a^2 + n \cdot a^2 = 0$$

$$X_1^2 + X_2^2 + X_3^2 + X_4^2 + \ldots . X_n^2 - 2(x_1 + x_2 + x_3 , \ldots\ldots , + x_n) \cdot a + n \cdot a^2 = 0$$

$$(X_1^2 - 2X_1 + a^2) + (X_2^2 - 2X_2 + a^2) + \ldots . . + (X_n^2 - 2X_n + a^2) = 0$$

$$(X_1 - a)^2 + (X_2 - a)^2 + \ldots . . + (X_n - a)^2 = 0$$

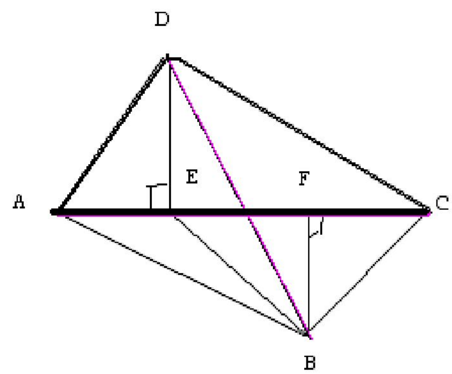
$$(X_1 - a)^2 = 0 \quad , \quad (X_2 - a)^2 = 0 , \quad \ldots . . , \quad (X_n - a)^2 = 0$$
$$(X_1 - a) = 0 \quad , \quad (X_2 - a) = 0 , \quad \ldots . . , \quad (X_n - a) = 0$$

$$x_1 = x_2 = x_3 = \ldots\ldots = x_n = a$$

4. Dreptunghiul ABCD cu latura DC = a şi BC = b (a > b) , se îndoaie de-a lungul diagonalei AC până ce planele ABC şi ADC devin perpendiculare . Să se calculeze lungimea segmentului BD astfel obținut .

(7 puncte)

<u>Soluție</u>



(ACD) $\perp$ (ABC)

Fie DE $\perp$ AC şi BF $\perp$ AC

AC = $\sqrt{a^2+b^2}$

Δ ACD $\equiv$ Δ ABC $\Rightarrow$ DE=BF

DE = $\dfrac{AD \cdot DC}{AC} = \dfrac{ab}{\sqrt{a^2+b^2}}$

AD^2 = CF$\cdot$ AC $\Rightarrow$ CF= $\dfrac{b^2}{\sqrt{a^2+b^2}}$

BC^2 = AE$\cdot$ AC $\Rightarrow$ AE= $\dfrac{b^2}{\sqrt{a^2+b^2}}$

EF= AC — (AE+FC) = $\dfrac{a^2-b^2}{\sqrt{a^2+b^2}}$

Δ EFB($\prec F = 90^0$) $\Rightarrow$ EB^2 = EF^2 +FB^2 $\Rightarrow$ EB$^2 = \dfrac{a^4-a^2b^2+b^4}{a^2+b^2}$

Δ DEB($\prec E = 90^0$) $\Rightarrow$ DB^2 = ED^2 +EB^2 $\Rightarrow$ DB$^2 = \dfrac{a^4+b^4}{a^2+b^2}$ $\Rightarrow$

$\Rightarrow$ DB= $\sqrt{\dfrac{a^4+b^4}{a^2+b^2}}$

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